# Exponential Mean-Square Stability of Linear Systems subject to Slowly Varying Delays with Known Stochastic Distribution

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Abstract—Time delay is a critical aspect concerning stability and robustness of controlled systems. This paper considers a class of linear time-delayed systems where the distribution and the maximum change rate of the delay are known. For these systems, it proposes a method to investigate their stability. Therefore, the delay is partitioned into intervals with occurrence probabilities to approximate the delay distribution. The number of intervals can be freely chosen to trade off between complexity and the quality of the approximation of the distribution. Considering this delay distribution approximation, the system is analysed for exponential stability in mean-square sense (ESMSS), and the benefits of this method are shown in numeric examples. It is revealed that by better approximating the delay distribution the maximum allowable delay can be increased. Further, if the delay change rate bound gets small, the conservatism is reduced even more. The result of this stability analysis is a statement for the expected value of the states at infinite time. Thus, no statement about stability for short time frames is made, which needs to be considered when choosing this approach.

*Index Terms*—Delay distribution, slow varying delay, delay partitioning, Exponential Stability in Mean-Square Sense (ESMSS)

# I. INTRODUCTION

Time delay is a common problem in the practical implementation of controlled systems on distributed platforms. A well-known use case of systems containing delays is the field of network control, where a digital protocol is used to communicate between individual systems and controllers. An example of such a system is a steering system in a vehicle. The central control unit sends an input signal to the electrical actuator of the steering system. This actuator input is calculated based on data received from sensors via network communication. There is, therefore, a communication delay between the sensor and the controller, as well as between the controller and the actuator. Often, a delay, for example resulting from a communication, has a negative effect on the stability of the considered system and thus needs to be included in the stability analysis of the system. The methods for stability analyses of these systems can be separated into two main

categories: delay-dependent and delay-independent. Delayindependent stability methods are, in comparison, conservative and thus often lack the desired performance. Examples of delayindependent stability based on Lyapunov-Krasovskii (LK) and Lyapunov-Razumikhin (LR) approaches, as well as an overview of the topic, can be found in [1, Section 3.3]. Many different delay-dependent approaches for various system classes can be found in literature. These methods can be divided into constant and time-varying delay stability analysis methods. For constant delays, frequency- and time-domain methods can be used, whereas only the time-domain approaches are valid for systems containing time-varying-delays. In this work will only consider systems with a time-varying delay. A group of methods frequently used for this, incorporating the delay derivative, are based on LK-functionals including model transformation [2], [3], free weighting matrices [4], inequalities, namely inequalities based on the Jensen [5] and Wirtinger inequality [6], [7], and delay partitioning [8], [9] to approximate the derivative of the functional. The delay in these methods is limited by an upper and sometimes a lower bound not equal to zero, then called non-small delayed systems. If the delay is varying but has an upper bound on the change rate, it is referred to as a slow varying delay [10].

In practical applications, the delay can be analysed or observed over a period of time, and a stochastic delay distribution can be derived [11]. Using this information, exponential stability in mean square sense (ESMSS) can be applied to the system [12]. Based on a probability-based model, this approach investigates whether stability can be expected for the states of the system. Parts of the stochastic system weighted by their occurrence probability may exhibit unstable behavior, for example, resulting from high delays. However, the system is still regarded as ESMSS when these parts are stabilized by the stable components of the stochastic system weighted by their occurrence probability. The conclusion is that the expected values of the states are stable [11]. A similar approach was used in [11] and [13], where a system containing a low nominal delay is analysed for ESMSS, which only for some exceptional cases contains long delay peaks. The work in [14] introduces

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a freely chosen number of intervals that partition the delay. These intervals can be used to approximate different delay distributions. Increasing the number of intervals approximates the distribution better, but at the cost of complexity.

This work will consider a system with a slow varying delay with a known upper bound on the change rate and a known delay distribution. This combination of delay information was not studied yet. Our work will use this information to reduce conservatism when analysing the stability of the system. Therefore, a theorem to prove ESMSS is derived and tested on numeric examples. Furthermore, the approach compared with the related work [14] reduces some further conservatism by better approximating the double integral term used in the LK-functional applying a Wirtinger-based inequality.

The paper is structured as follows: First, the system and the probability model, some useful lemmas and definitions will be presented. A linear matrix inequality (LMI) and its proof are shown in the main part to analyse ESMSS for the defined system. Numerical examples, a discussion, and a conclusion will close this paper.

Notations: Throughout this paper,  $\mathbb{R}^s$  denotes the *s* dimensional Euclidean space, and  $\mathbb{R}^{s \times l}$  is the set of real matrices with the dimensions  $s \times l$ . P > 0 (P < 0) means that *P* is a symmetric positive (negative) definite matrix,  $\mathbb{E}\{\cdot\}$  denotes expected value, and  $\lambda(\cdot)$  represents the eigenvalues. Indices max and min denote the maximum and minimum value of the considered vector. *i* is used as an index for sums, while *j* and *k* reference positions in matrices. For state vectors  $x(t) = x_t$  and  $x(t - \tau(t)) = x_t(-\tau(t))$  are adapted from the literature.

#### II. BACKGROUND AND STOCHASTIC DELAY FRAMEWORK

Consider a system defined by

$$\dot{x}_t = Ax_t + A_d x_t (-\tau(t)), \tag{1}$$

$$x_t = \Psi(t), \ t \in [-\tau_M, 0], \tag{2}$$

where  $x_t \in \mathbb{R}^s$  represents the state vector and the system matrices A,  $A_d$  are known and have appropriate dimensions. Equation (2) defines the initial condition of the system.  $\tau(t)$ represents the varying delay in the bounds of  $0 \leq \tau(t) \leq \tau_M$ and  $\dot{\tau}(t) \leq d_M < 1$ .

**Remark 1.** The upper bound of the upper bound  $d_M$ , that is 1, can be regarded as the information arriving at  $A_d$  in the right order. Further  $A_d$  in the autonomous system (1) can also be thought of as BK, with input matrix B and controller gain K, for a closed loop delayed state feedback system.

Further, a model including the distributed delay using Bernoulli distributed sequences will be defined. Let us define N intervals with the bounds

$$0 = \bar{\tau}_0 > \bar{\tau}_1 > \dots > \bar{\tau}_{N-1} > \bar{\tau}_N = \tau_M \tag{3}$$

to represent the entire rage of  $\tau(t)$ . Defining N sets:

$$\sigma_i = \{t | \tau(t) \in [\bar{\tau}_{i-1}, \bar{\tau}_i)\}. \tag{4}$$

Based on these sets, the stochastic Bernoulli sequence

$$\beta_i(t) = \begin{cases} 1, & \tau(t) \in \sigma_i \\ 0, & \tau(t) \notin \sigma_i \end{cases}$$
(5)

and analogous a Bernoulli-like delay variable  $\tau_i(t)$ 

$$\tau_i(t) = \begin{cases} \tau(t), & \tau(t) \in \sigma_i \\ 0, & \tau(t) \notin \sigma_i \end{cases}$$
(6)

are defined. With the expected value of  $\beta_i(t)$  defined as

$$\operatorname{Prob}(\beta_i(t) = 1) = \mathbb{E}\{\beta_i(t)\} = \beta_i.$$
(7)

**Remark 2.**  $\bar{\tau}_i$  is introduced for the constant interval bounds to clearly differentiate from changing values like  $\tau_1(t), ... \tau_N(t)$ . Further  $\beta_i^2(t) = \beta_i(t)$  and  $\beta_i(t)\beta_j(t) = 0$  for  $i \neq j$  are valid.

With this definition, the system (1) is also represented by

$$\dot{x}_{t} = Ax_{t} + \sum_{i=1}^{N} \beta_{i}(t) A_{d}x_{t}(-\tau_{i}(t))$$
(8)

with initial condition (2) and including the distribution of the delay in the model. Further, two lemmas and a remark will be given, which will be used in the contribution of this paper. The first lemma defines ESMSS for the previously defined class of systems and will be used to prove that the theorem used leads to this kind of stability condition.

**Lemma 3.** *ESMSS [11]:* A system defined by (8) with initial condition (2) said to be exponential stable in mean-square sense if there exist  $\alpha, \theta > 0$  such that

$$\mathbb{E}\{\|x_t\|^2\} \leqslant \alpha e^{-\theta t} \sup_{-\bar{\tau}_M \le \Psi \le 0} \mathbb{E}\{\|x(\Psi)\|^2 + \|x(\dot{\Psi})\|^2\}.$$
 (9)

The second lemma given is an integral inequality. This inequality is used to approximate the integrals containing derivatives of the state and thus approximate the LK-functional derivative in a way convertible to an LMI representation.

**Lemma 4.** Wirtinger-Inequality [6]: For every continuous function  $\omega$  in  $[a, b] \rightarrow \mathbb{R}^s$  and R > 0 the following inequality holds:

$$\int_{a}^{b} \dot{\omega}^{T}(u) R \dot{\omega}(u) du \geq \frac{1}{b-a} (\omega(b) - \omega(a))^{T} R (\omega(b) - \omega(a)) + \frac{3}{b-a} \Omega^{T} R \Omega,$$
(10)
with  $\Omega = \left[ \omega(a) + \omega(b) - \frac{2}{b-a} \int_{a}^{b} \omega(a) du \right]$ 

with 
$$\Omega = \left[\omega(a) + \omega(b) - \frac{2}{b-a} \int_a^b \omega(u) du\right].$$

Closing this section, a remark will be given to make handling inequalities in the proof easier by emphasizing eigenvalue-based worst- and best-case assumptions.

**Remark 5.** Optimization principle for eigenvalues [15]: Let  $M \in \mathbb{R}^{n \times n}$  be an arbitrary symmetric matrix with  $\lambda_{\min}(M)$  the smallest and  $\lambda_{\max}(M)$  the largest eigenvalue of M. Then for an arbitrary vector  $x \in \mathbb{R}^n$  the following holds

$$\lambda_{\min}(M)x^T x \leqslant x^T M x \leqslant \lambda_{\max}(M)x^T x.$$
(11)

## **III. DELAY DISTRIBUTION-DEPENDENT ESMSS**

The contribution of this paper is to reduce conservatism in the stability analysis for systems when more information than only the bounds of the delay are known. More specifically, in the considered approach the information about the stochastic distribution and the maximum change rate of the delay are assumed to be known. Using this information, it can be analysed that for a system, where the delay is partitioned into intervals with an occurrence probability, whether the expected value of the states of the system are stable or unstable. For this, in the following, an LMI is constructed to test systems defined as in (8) for ESMSS.

**Theorem 6.** Considering a system defined as in (8) with initial condition (2), a stochastic partitioned delay into N intervals according to (3) to (6), and an upper bound of  $d_M$  for the delay changing rate. This system is ESMSS if there exists a set of symmetric matrices  $P_i$ ,  $Q_i$ ,  $S_i$ ,  $R_i > 0$  for i = 1, ..., N and  $R_i^c > 0$  for i = 2, ..., N for which (12) is satisfied:

$$\Pi = \begin{bmatrix} \pi_{1,1} & \pi_{2,1}^T & \pi_{3,1}^T \\ \pi_{2,1} & \pi_{2,2} & \pi_{3,2}^T \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} \end{bmatrix} < 0,$$
(12)

The  $Q_i, S_i, R_i, R_i^c \in \mathbb{R}^{n \times n}$  with n the number of states of (8) whereas the matrix  $P_i \in \mathbb{R}^{2n \times 2n}$  is defined as:

$$P_{i} = \begin{bmatrix} P_{i,1} & P_{i,2} \\ P_{i,2} & P_{i,3} \end{bmatrix}.$$
 (13)

The construction of the  $\Pi$  matrix in (12) depends on the system, the partitioning and distribution of the delay, and the matrices containing the decision variables. The sub matrices  $\pi_{1,1}$ ,  $\pi_{2,1}$ ,  $\pi_{3,1}$ ,  $\pi_{2,2}$ ,  $\pi_{3,2}$ , and  $\pi_{3,3}$  are defined in the appendix for readability reasons.

*Proof.* We choose the following LK-functional close to [16] and use the summation idea of [14]:

$$V(x_{t}) = \sum_{i=1}^{4} V_{i}(x_{t}),$$

$$V_{1}(x_{t}) = \sum_{i=1}^{N} \bar{x}_{t}^{T} P_{i} \bar{x}_{t},$$

$$V_{2}(x_{t}) = \sum_{i=1}^{N} \int_{t-\bar{\tau}_{i}}^{t} x^{T}(s) S_{i} x(s) ds,$$

$$V_{3}(x_{t}) = \sum_{i=1}^{N} \int_{t-\tau(t)}^{t-\bar{\tau}_{i-1}} x^{T}(s) Q_{i} x(s) ds,$$

$$V_{4}(x_{t}) = \sum_{i=1}^{N} \int_{t-\bar{\tau}_{i}(t)}^{t-\bar{\tau}_{i-1}} \int_{s}^{t} \dot{x}^{T}(v) R_{i} \dot{x}(v) dv ds$$

$$+ \sum_{i=2}^{N} \int_{t-\bar{\tau}_{i-1}}^{t} \int_{s}^{t} \dot{x}^{T}(v) R_{i} \dot{x}(v) dv ds,$$
(14)

with  $\bar{x}_t = \left[x_t^T, \int_{t-\tau_i}^t x(s) ds^T\right]^T$ . Clearly  $V(x_t) > 0$  with

 $P_i, S_i, Q_i, R_i, R_i^c > 0$ . The derivative of term V1 and V2 is straightforward

$$\dot{V}_1(x_t) = \sum_{i=1}^N \bar{x}_t^T P_i \dot{\bar{x}}_t + \dot{\bar{x}}_t^T P_i \bar{x}_t, \qquad (15)$$

$$\dot{V}_2(x_t) = \sum_{i=1}^{N} \left[ x_t^T S_i x_t - x_t^T (-\bar{\tau}_i) S_i x_t (-\bar{\tau}_i) \right].$$
(16)

The derivative of the variable integral terms  $V_3(x_t)$  and  $V_4(x_t)$ , using the terms

$$\Delta_R = \sum_{i=1}^{N} (\bar{\tau}_i - \bar{\tau}_{i-1}) R_i + \sum_{i=2}^{N} \bar{\tau}_{i-1} R_i^c, \qquad (17)$$

$$\Delta_d = (1 - d_M),\tag{18}$$

can be expressed by the following derivatives:

$$\dot{V}_{3}(x_{t}) = \sum_{i=1}^{N} \left[ x_{t}^{T} Q_{i} x_{t} - \Delta_{d} x_{t}^{T} (-\bar{\tau}_{i}(t)) Q_{i} x_{t} (-\bar{\tau}_{i}(t)) \right],$$
(19)

$$\dot{V}_{4}(x_{t}) = \dot{x_{t}}^{T} \Delta_{R} \dot{x}_{t} - \Delta_{d} \sum_{i=1}^{N} \int_{t-\bar{\tau}_{i}}^{t-\bar{\tau}_{i-1}} \dot{x}^{T}(s) R_{i} \dot{x}(s) ds - \sum_{i=2}^{N} \int_{t-\bar{\tau}_{i-1}}^{t} \dot{x}^{T}(s) R_{i}^{c} \dot{x}(s) ds,$$
(20)

where the second and third term of (20) will be estimated with lemma 4 resulting in

$$\int_{t-\bar{\tau}_{i}}^{t-\bar{\tau}_{i-1}} \dot{x}^{T}(u) R_{i} \dot{x}(u) du \geq \zeta \Omega^{T} \begin{bmatrix} 4R_{i} & 2R_{i} & -6\zeta R_{i} \\ 2R_{i} & 4R_{i} & -6\zeta R_{i} \\ -6\zeta R_{i} & -6\zeta R_{i} & 12\zeta^{2}R_{i} \end{bmatrix} \Omega,$$

$$(21)$$

with

$$\Omega = [x_t^T(-\bar{\tau}_{i-1}), x_t^T(-\bar{\tau}_i), \int_{t-\bar{\tau}_i}^{t-\bar{\tau}_{i-1}} x(u) du^T]^T$$

and  $\zeta = (\bar{\tau}_i - \bar{\tau}_{i-1})^{-1}$ . This analogously is applied to the third term of (20). Equations (15) to (21) can be combined to the following LMI

$$\tilde{x}_t^T \Pi \tilde{x}_t < 0 \tag{22}$$

using  $\Pi$  from equation (12) and replacing  $\beta_i$  by  $\beta_i(t)$ . The vector  $\tilde{x}_t$  thereby is defined as

$$\tilde{x}_{t} = \left[x_{t}^{T}, x_{t}^{T}(-\tau_{1}(t)), ..., x_{t}^{T}(-\tau_{N}(t)), x_{t}^{T}(-\bar{\tau}_{1}), ..., x_{t}^{T}(-\bar{\tau}_{N}), \int_{t-\bar{\tau}_{1}}^{t} x(s)ds^{T}, ..., \int_{t-\bar{\tau}_{N}}^{t-\bar{\tau}_{N-1}} x(s)ds^{T}\right]^{T}$$
(23)

**Remark 7.** Therefore, integral terms are represented as the sum of multiple elements of  $\tilde{x}_t$ . Further, because the problem is convex, only the vertices of the considered problem need to be analysed.

Taking the expected value of (22) leads to equation (12). The following proof shows that equation (12) proves that system (8) is ESMSS. It follows the lines of the proof of theorem 1 in [17]. Defining a function  $W(x_t) = e^{\varepsilon t}V(x_t)$  for  $t \ge 0$  with a positive scalar  $\varepsilon$ . The derivative of this function is

$$\dot{W}(x_t) = \varepsilon e^{\varepsilon t} V(x_t) + e^{\varepsilon t} \dot{V}(x_t).$$
(24)

Taking the expected value this can be converted to

$$\mathbb{E}\{W(x_t)\} - \mathbb{E}\{W(x_0)\} = \int_0^t [\varepsilon e^{\varepsilon s} \mathbb{E}\{V(x(s))\} + e^{\varepsilon s} \mathbb{E}\{\dot{V}(x(s))\}] ds.$$
(25)

Equation (12) and (14) define  $\dot{V}(x_t) < 0$  and  $V(x_t) > 0$ , so it is easy to show that there exists a small enough  $\varepsilon$  to make the right side of this equation negative depending on the eigenvalues of  $P_i, S_i, Q_i, R_i, R_i^c$  in (14) and  $\Lambda = \lambda_{\max}\{\Pi\}$ [11]. From this and the left side of (25), it can be deduced that

$$\mathbb{E}\{V(x_t)\}e^{\varepsilon t} \leqslant \mathbb{E}\{V(x_0)\}\tag{26}$$

for this small  $\varepsilon$ . Replacing  $x_0$  in this equation with the initial conditions defined in (2) for the system defined in (8) and taking the worst case from remark 5

$$\mathbb{E}\{V(x_t)\}e^{\varepsilon t} \leqslant \sigma_1 \sup_{\substack{-\tau_M \leqslant s \leqslant 0\\ -\tau_M \leqslant s \leqslant 0}} \mathbb{E}\{\|\Psi(s)\|^2\} + \sigma_2 \sup_{\substack{-\tau_M \leqslant s \leqslant 0\\ -\tau_M \leqslant s \leqslant 0}} \mathbb{E}\{\|\dot{\Psi}(s)\|^2\},$$
(27)

$$\sigma_{1} = \sum_{i=1}^{N} \lambda_{\max}(P_{i}) + \sum_{i=1}^{N} [\bar{\tau}_{i}\lambda_{\max}(S_{i})] + \sum_{i=1}^{N} [(\bar{\tau}_{i} - \bar{\tau}_{i-1})\lambda_{\max}(Q_{i})],$$
  
$$\sigma_{2} = \sum_{i=1}^{N} [(\bar{\tau}_{i} - \bar{\tau}_{i-1})^{2}\lambda_{\max}(R_{i})] + \sum_{i=2}^{N} [(\bar{\tau}_{i-1})^{2}\lambda_{\max}(R_{i}^{c})].$$

Using the maximum of the scaling factors as worst case assumption  $\alpha = \max(\sigma_1, \sigma_2)$  and considering  $P_i, Q_i, R_i, R_i^c$  and  $S_i$  are positive definite the LK-functional can be lower bound by  $V(x_t) \ge \sum_{i=1}^N \lambda_{\min}(P_{i,1})x_t^T x_t$ . With this and the lemma 3, this proof is completed by

$$\mathbb{E}\{x_t^T x_t\} \leqslant \bar{\alpha} e^{-\varepsilon t} \sup_{-\tau_M \leqslant s \leqslant 0} \mathbb{E}\{\|\Psi(s)\|^2 + \|\dot{\Psi}(s)\|^2\}, \quad (28)$$

with 
$$\bar{\alpha} = \alpha / \sum_{i=1}^{N} \lambda_{\min}(P_{i,1}).$$

## IV. NUMERICAL EXAMPLES

To show the advantage of the proposed method for stability analysis considering time delays, a system will be analysed by varying the number of intervals defined to approximate the delay distribution and varying the upper bound on the change rate of the delay  $d_M$ . We use the example system from [6] and [14] defined as in (8) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$
 (29)

For simplicity, the delay is assumed to have a symmetric triangular distribution rising from zero to  $\frac{\tau_M}{2}$  and declining back to zero at  $\tau_M$ . The intervals are assumed to partition the delay into equal parts and therefore, are scaled by the delay maximum  $\tau_M$ . Table I displays the used distribution to approximate the distribution of the example. For each

Table I Occurrence Probability of the Intervals with Different Numbers of Equidistant Partitions

Ν	Occurrence probability [%]			
1	100			
2	50		50	
3	22	5	6	22
4	12.5	37.5	37.5	12.5

partitioning, the maximum allowable delay  $\tau_M$  is calculated for varying  $d_M$ . Fig. 1 illustrates this for the intervals and probabilities shown in Table I. The upper bound for the delay increases with an increasing number of intervals, which is the expected result because by better approximating the distribution the probability that the delay has the maximum interval value decreases. It also becomes evident that a lower upper bound on the change rate  $d_M$  also leads to a higher admissible delay for stability, which increases even faster, approaching zero. But, as will be discussed in the next section, with a decrease in the change rate and thus also with its upper bound decreasing, the idea of ESMSS is restricted. It reaches its maximum admissible delay at  $d_M = 0$  referring back to constant delays, and thus ESMSS does not apply to the problem. For this reason Fig. 1 shows the interval of  $d_M$  for which this approach is applicable. The case N = 2 can be compared to [11, Example 2] with  $\beta_0 = 0.5$ . With their suggested method  $d_M = 1$  and thus results for high delay change rates in less conservatism but for  $d_M < 0.57$  our approach results in a higher admissible delay.



Figure 1. Maximum admissible delay  $\tau_M$  for different numbers of partitions N and varying delay change rate bound  $d_M$ .



Figure 2. Maximum admissible delay  $\tau_M$  for different occurrence probabilities  $\beta_2, \beta_3$  and different upper bounds  $d_M$ .

To illustrate the advantages of the method presented using a second example, the number of partitions is kept constant, at N = 4, in the following. Instead, the stochastic occurrence probabilities of the delay intervals are varied over the different bounds of the change rate. The delay is again partitioned into equally large intervals with the constant occurrence probabilities  $\beta_1 = 0\%$  and  $\beta_4 = 35\%$ . The probabilities  $\beta_2$  and  $\beta_3$  are varied. Thereby,  $\beta_2$  starts at a 65% probability and ends at 0%. Following this,  $\beta_3$  has a probability of  $65\% - \beta_2$ . Considering Fig. 2, the system results in a higher maximum stable delay for a higher mean delay when  $\beta_3$  gets larger. This result is analogous to the literature, e.g. [1, Section 3.7], where increasing the lower bound of a varying delay also increases the maximum admissible upper bound of the system. In this example, this can be regarded as the expected value delay width.

## V. DISCUSSION AND CONCLUSION

In the previous sections a method to analyse stability for a system containing a delay was presented. As innovation, the known information of the upper bound of the delay change rate was combined with the stochastic distribution of the delay to reduce the conservatism and thus, increase the maximum admissible delay. So systems analysed as unstable by other methods can show stability under this method using the delay information. This improvement depends on the number of separated delay intervals, when stochastically modelling the system, and the delay change rate upper bound. For example considering the steering system mentioned in the introduction, the communication is mainly defined by safety requirements. Because of that the communication is simple, thus has a limited number of influence parameters, and the latency variation is mainly defined by a slow clock drift. Thus the distribution is measurable and the change rate is low. So this method results in a non-conservative stability statment for this example. A clear drawback compared to other stability analyses methods is the complexity and the number of decision variables, which is  $Nn^2 + (6N-1)\sum_{i=1}^{n} i$ , with N being the number of intervals and n the dimension of the considered system. To reduce the this complexity a simpler LK-funcitonal, e.g. with out the

double integral terms, can be chosen but by this modification the method includes more conservatism.

This analysis can not be translated directly to real applications. When implementing this approach, the properties of the system and the control target must be considered. Limiting the derivative of the delay restricts the idea of an even occurrence of the probability distribution. Thus, the stability for the analysed system at infinite time is given, but for short time frames no statements about stability nor performance can be concluded. For example, considering a finite time frame, a system considered ESMSS can start oscillating with increasing magnitude and become stable again at a later time frame. If these things are considered, the proposed method results in non-conservative stability margins.

Further research topics deduced from this paper are how the LK-functional can be chosen to enhance the performance or to minimize the complexity and the number of decision variables. This is a very important question because it is crucial considering this when implementing the purposed approach. Furthermore, a question is how small the lower bound of the change rate can be set to still have the desired performance for short periods of time. The question is also whether and to what extent this methodology can be applied to discrete-time systems.

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#### APPENDIX

## CONSTRUCTION OF THE THEOREM MATRIX

In the appendix, the equations to build the matrix used in theorem 6 are shown. The matrix and its dimensions depend on the number of delay partitions, with there occurrence probabilities, the system matrices, and the decision variable matrices. The  $\Pi$  matrix in (12) can be build by the following sub-matrices:

$$\pi_{1,1} = \begin{bmatrix} \psi_{1,1} & \psi_{2,1}^{T} & \psi_{3,1}^{T} & \dots & \psi_{N+1,1}^{T} \\ \psi_{2,1} & \psi_{2,2} & 0 & \dots & 0 \\ \psi_{3,1} & 0 & \psi_{2,2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \psi_{N+1,1} & 0 & \dots & 0 & \psi_{N+1,N+1} \end{bmatrix}, \quad (30)$$

$$\psi_{1,1} = \sum_{i=1}^{N} (P_{i,1}A + A^{T}P_{i,1} + P_{i,2}) + \sum_{i=1}^{N} S_{i} + Q_{1}$$

$$- \Delta_{d} \frac{4R_{1}}{\bar{\tau}_{1}} - \sum_{i=2}^{N} \frac{4R_{i}^{c}}{\bar{\tau}_{i-1}} + A\Delta_{R}A,$$

$$\psi_{j+1,1} = \beta_{j}A_{d}^{T} \left[ \sum_{i=1}^{N} P_{i,1} + \Delta_{R}A \right], \quad j = 1, \dots, N,$$

$$\psi_{j+1,j+1} = \beta_{j}A_{d}\Delta_{R}A_{d} - \Delta_{d}Q_{j}, \quad j = 1, \dots, N,$$

$$\pi_{2,1} = \begin{bmatrix} \gamma_{1,1} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \gamma_{N,1} & 0 & \dots & 0 \end{bmatrix}, \quad (31)$$

$$\gamma_{1,1} = -2\frac{R_{2}^{c} + \Delta_{d}R_{1}}{\bar{\tau}_{1}} - P_{2,1},$$

$$\gamma_{i,1} = -\frac{2R_{i+1}^{c}}{\bar{\tau}_{i}} - P_{i,2}, \quad i = 2, \dots, N-1,$$

$$\gamma_{N,1} = -P_{N,2},$$

$$\pi_{3,1} = \begin{bmatrix} \rho_{1,1} & \dots & \rho_{1,N+1} \\ \vdots & \ddots & \vdots \\ \rho_{N,1} & \dots & \rho_{N,N+1} \end{bmatrix}, \quad (32)$$

$$\begin{split} \rho_{1,1} &= \sum_{i=1}^{N} \left[ P_{i,2}A + P_{i,3} + \frac{6R_{i+1}^{i}}{\bar{\tau}_{i}^{2}} \right] + \Delta_{d} \frac{6R_{1}}{\bar{\tau}_{1}^{2}}, \\ \rho_{j,1} &= \sum_{i=j}^{N} \left[ P_{i,2}A + P_{i,3} + \frac{6R_{i+1}^{i}}{\bar{\tau}_{i}^{2}} \right], \ j = 2, \dots, N-1, \\ \rho_{N,1} &= P_{N,2}A + P_{N,3}, \\ \rho_{j,k+1} &= \beta_{s} \sum_{i=j}^{N} P_{i,2}A_{d}, \ j = 1, \dots, N \text{ and } k = 1, \dots, N, \\ \pi_{2,2} &= \begin{bmatrix} \omega_{1,1} & \omega_{2,1}^{T} & 0 & \dots & 0 \\ \omega_{2,1} & \omega_{2,2} & \omega_{3,2}^{T} & \ddots & \vdots \\ 0 & \omega_{3,2} & \omega_{3,3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \omega_{N,N-1}^{T} \\ 0 & \dots & 0 & \omega_{N,N-1} & \omega_{N,N} \end{bmatrix}, \\ (33) \\ \omega_{j,j} &= -\frac{4\Delta_{d}R_{j}}{\bar{\tau}_{j} - \bar{\tau}_{j-1}} - \frac{4\Delta_{d}R_{j+1}}{\bar{\tau}_{j+1} - \bar{\tau}_{j}} - \frac{4R_{j+1}^{e}}{\bar{\tau}_{j}} + Q_{j+1} - S_{j}, \\ j = 1, \dots, N-1, \\ \omega_{N,N} &= -\frac{4\Delta_{d}R_{N}}{\bar{\tau}_{N} - \bar{\tau}_{N-1}} - S_{N}, \\ \omega_{j+1,j} &= -\frac{2\Delta_{d}R_{j+1}}{\bar{\tau}_{j+1} - \bar{\tau}_{j}}, \ j = 1, \dots, N-1, \\ \omega_{N,N} &= -\frac{4\Delta_{d}R_{N}}{\bar{\tau}_{N} - \bar{\tau}_{N-1}} - S_{N}, \\ \omega_{j+1,j} &= -\frac{2\Delta_{d}R_{j+1}}{\bar{\tau}_{j+1} - \bar{\tau}_{j}}, \ j = 1, \dots, N-1, \\ \omega_{N,N} &= -\frac{4\Delta_{d}R_{N}}{\bar{\tau}_{N} - \bar{\tau}_{N-1}} - S_{N}, \\ \omega_{j+1,j} &= -\frac{2\Delta_{d}R_{j+1}}{\bar{\tau}_{j+1} - \bar{\tau}_{j}}, \ j = 1, \dots, N-1, \\ \phi_{N,N} &= \frac{6\Delta_{d}R_{j}}{(\bar{\tau}_{j} - \bar{\tau}_{j-1})^{2}} + \frac{6R_{j+1}^{e}}{\bar{\tau}_{j}^{2}} - P_{3,j}, \ j = 1, \dots, N-1, \\ \Phi_{j,k} &= \frac{6R_{k+1}^{e}}{\bar{\tau}_{k}^{2}} - P_{3,k}, \ s = 2, \dots, N \ \text{and} \ j < k, \\ \Phi_{j,j} &= \frac{6\Delta_{d}R_{j}}{(\bar{\tau}_{j} - \bar{\tau}_{j-1})^{2}} + \frac{6R_{j+1}^{e}}{\bar{\tau}_{k}^{2}} - P_{3,N} \\ \Phi_{j,j-1} &= \frac{6\Delta_{d}R_{j}}{(\bar{\tau}_{j} - \bar{\tau}_{j-1})^{2}} - P_{3,N} \\ \Phi_{j,j-1} &= \frac{6\Delta_{d}R_{j}}{(\bar{\tau}_{j} - \bar{\tau}_{j-1})^{2}}, \ j = 2, \dots, N \ \text{and} \ k < j, \\ \upsilon_{j,k} &= -\sum_{i=j}^{N} \frac{12R_{i+1}^{e}}{\bar{\tau}_{i}^{3}} - \frac{12\Delta_{d}R_{j}}{(\bar{\tau}_{i} - \bar{\tau}_{i-1})^{3}}, \ j = 1, \dots, N-1, \\ \upsilon_{N,N} &= -\frac{12\Delta_{d}R_{N}}{(\bar{\tau}_{N} - \bar{\tau}_{N-1})^{3}}. \end{cases}$$