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Optimal 2nd Order LTI System Identification

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***Abstract*— Just as optimal control addresses the inexact science of selecting controller gains, optimal system identification balances the effects of linearization, estimation and order reduction, to obtain the "best fit" approximation of a target electrical, mechanical, or electro-mechanical system. Like any engineering design problem, it involves matching a set of free design parameters to a requirement specification that defines what "best" means. In this paper, closed-form metrics of a normalized second-order system are used to develop a clear and simple design process to identify a 2nd order approximation that exhibits the most relevant dynamic characteristics of the target system. The process identifies the optimal parameters of an under or over-damped system from its step-response, and refines the approximation using its impulse-response. The approach is formulaic, non-iterative, and may be used to fit a second-order approximation to a higher-order system response, without the need for a complex search algorithm.**

I. Introduction

Any practical controller design project begins with system identification of each component, or sub-system. When a sub-system is outsourced, full information is unavailable so an approximation is needed. A 2nd order approximation with the dominant dynamics and DC gain is often sufficient since non-linearities which are present but cannot be included in an LTI model, are often more significant than the deviation from a 2nd order approximation. When detailed model information is available, a 2nd order approximation may still be preferable to simplify the controller design process.

Closed form equations to identify the dynamic behavior of under-damped 2nd order systems are well known (see for example [1, 2]) and have recently been extended to include over-damped systems [3] using their inflection point. These are useful when the target system is 2nd order, but do not adequately address deviations such as higher-order systems or non-linearities. Iterative methods have been developed to reduce the order of higher-order systems (see the review in [4]), but they are difficult to implement and are often beyond the scope of a novice engineer or the time budget of an engineer developing a controller for a non-critical application.

Here, a redundant set of closed form equations that each consider a different aspect of an under-damped or over-damped system from its step or impulse response, are combined to provide an approximation that targets the metrics of greatest importance. The process may be used to identify a black-box system from its step response which is easily obtained in the lab or available in a data-sheet, or to reduce the order of a white-box system whose impulse response may be mathematically generated.

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The paper is organized as follows. Section II identifies an under-damped step-response with redundant equations and applies a weighted sum to arrive at an optimal estimation with respect to a practical requirement specification. It then identifies an over-damped step-response and indicates when a 1st order approximation is preferable. Section III repeats the process for an impulse-response. Section IV shows a practical example of each system identification scenario. Section V provides concluding remarks.

II. Step Response Identification

A. Under-damped Systems

In [1], equations (1-2) are developed for Peak time (T_p) and Settle time (T_s) as a function of damping coefficient (ζ) and natural frequency (ω_n). Substituting T_p into the time domain representation of the step response of a generalized 2nd order system produces the closed-form relationship between ζ and Percentage Overshoot (%OS) shown in (3).

$$T_p = \frac{\pi}{\omega_n \beta}; \quad \beta = \sqrt{1-\zeta^2} \tag{1}$$

$$T_s = \frac{4}{\omega_n \zeta} \tag{2}$$

$$\zeta = \frac{-\ln\left(^{\%OS}/_{100}\right)}{\sqrt{\pi^2 + \ln^2\left(^{\%OS}/_{100}\right)}} \tag{3}$$

As claimed in [1], "A precise analytical relationship between rise time and damping ratio, ζ, cannot be found", but it is shown here that a precise analytical relationship between Rise Time (T_r), ζ and ω_n can be found. The inverse Laplace Transform of the step response of the generalized 2nd order equation (4) is shown in (5).

$$\frac{1}{s}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{4}$$

$$1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin\left(\beta\omega_n t + \tan^{-1}\left(^{\beta}/_{\zeta}\right)\right) \tag{5}$$

T_r is the minimum time when the negative term in (5) equals 0, which is either when the exponential or the sine function equal 0. The exponential is 0 when t = ∞ (a trivial result), so T_r is the shortest time that sin(*arg*)=0, which occurs any time *arg* is an integer multiple of π (*arg*=nπ). Setting n=0 results in a negative T_r. Setting n=1 produces the solution to T_r shown in (6-7).

$$\beta\omega_n T_r + \tan^{-1}\left(^{\beta}/_{\zeta}\right) = \pi \tag{6}$$

$$T_r = \frac{1}{\beta\omega_n}\pi - \tan^{-1}\left(^{\beta}/_{\zeta}\right) \tag{7}$$

Since (1-2) and (7) each relate a different time metric to ζ and ω_n, they can each be rearranged to solve for ω_n as a function of ζ and the individual time metric (8-10).

$$\omega_{nr} = \frac{1}{\beta T_r}\pi - \tan^{-1}\left(\beta / \zeta\right) \quad (8)$$

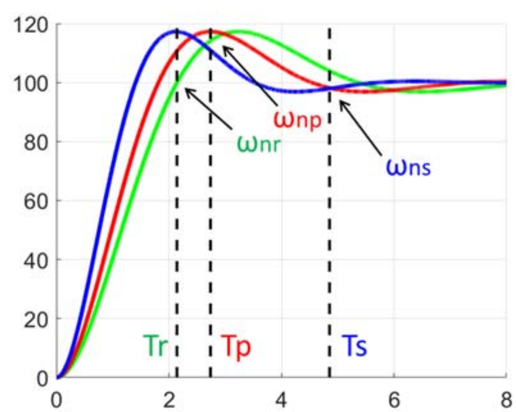
$$\omega_{np} = \frac{\pi}{T_p \beta} \quad (9)$$

$$\omega_{ns} = \frac{4}{T_s \zeta} \quad (10)$$

This provides three redundant methods for identifying a 2nd order step response. ζ is first obtained (3) to provide the desired %OS, and then combined with a desired metric (T_r, T_p, T_s) to generate the corresponding ω_n (8-10). The resulting model has the exact T_r or T_p when (8-9) are used, or the approximate T_s when (10) is used since the phase of the sine term is ignored in the derivation of (2). For example, Fig. 1 contains three curves generated using a common ζ with 20% Overshoot, and a common DC Gain of 100.

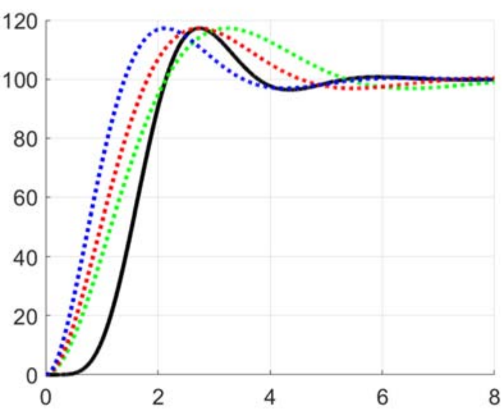
- Green: ω_{nr} computed from (8) with T_r=2.1
- Red: ω_{np} computed from (9) with T_p=2.7
- Blue: ω_{ns} computed from (10) with T_s=4.9

Figure 1. Derived 2nd Order Step Responses.



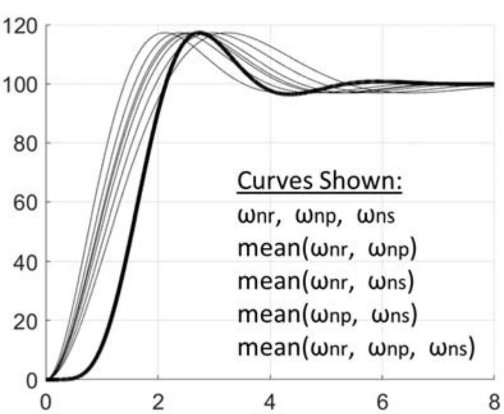
Natural frequency ω_n becomes a free design parameter to generate the 2nd order approximation that best satisfies a requirement specification. When the target system is a 2nd order system, all three curves are equivalent and equal to the target. A 6th order target step response is shown (black curve) in Fig. 2 with three approximations, each corresponding to one of (8-10). The target curve has the same T_r as the green curve, the same T_p as the red curve, the same T_p as the blue curve, and 20% OS, just like all three approximations.

Figure 2. 2nd Order Approximations of Black-Box Step Response.



The choice of approximation is based on the relative importance of the individual metrics, or a compromise can be developed. Since each curve is generated from single design parameter ω_n, a weighted sum can be used. In Fig. 3, the target step-response and various approximations using mean values are shown. Of course, mean values correspond to equal weighting coefficients but any weighting coefficients could be chosen.

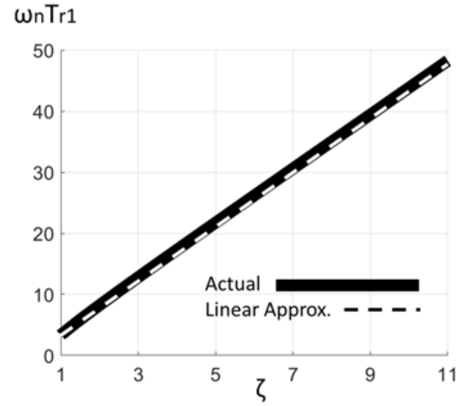
Figure 3. 2nd Order Approximations from Weighted Sums.



B. Over-damped Systems

For an over-damped step response, neither T_r nor T_p exist since the system never reaches final value. Instead, modified rise time T_{r1}, the time to rise from 10% to 90% of final value, is used. Although a closed-form relationship between ζ, ω_n, and T_{r1} is not known, it is shown in [1] that the linear relationship shown in Fig. 4 exists between ζ and $\omega_n T_{r1}$, which can be estimated by (11).

Figure 4. Modified Rise Time of Over-Damped Step Response.



$$\omega_n T_{r1} = 4.44\xi - 1.15 \quad (11)$$

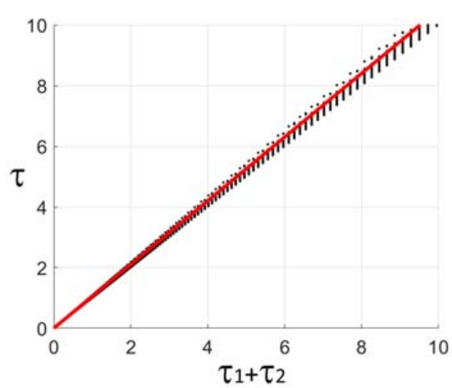
Since a 2nd order system has two parameters (ζ and ω_n), an additional metric is needed to complete the approximation. Although T_s is easily obtainable, equation (2) is not reliable since the poles of an overdamped 2nd order system do not share a common real component. Instead, the time constant τ, the time required to reach 63% of final value, which is also available by inspection, is used for this purpose. This metric (τ) does not appear explicitly in the definition of a 2nd order system with two real poles (12), but is shown in (13) to be the average of the two individual time constants which may be represented in terms of $\omega_n\tau$ using a linear approximation (14) taken from Fig. 5. In Fig. 5, the point cloud from computing τ and $\tau_1+\tau_2$ for a wide range of ζ and ω_n values shows a close linear correspondence between them.

$$\frac{\left(\frac{1}{\tau_1\tau_2}\right)}{s^2+\left(\frac{\tau_1+\tau_2}{\tau_1\tau_2}\right)s+\left(\frac{1}{\tau_1\tau_2}\right)}; \quad \tau_1 = \frac{-1}{p_1} \quad \tau_2 = \frac{-1}{p_2} \quad (12)$$

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}; \quad \zeta = \omega_n\frac{(\tau_1+\tau_2)}{2} \approx {\omega_n\tau}/{2.1} \quad (13)$$

$$\tau \approx 1.05(\tau_1+\tau_2) \quad (14)$$

Figure 5. Linear Approximation of 2nd Order Time Constant.



Substituting (13) into (11) provides an estimate of ζ that is independent of ω_n and only relies on T_{r1} and τ that are read directly from the step-response. With ζ known, (13) is used to solve for ω_n (16).

$$\zeta = \frac{\tau}{3.86\,\tau - 1.83\,T_{r1}} \quad (15)$$

$$\omega_n = {2.1\zeta}/{\tau} \quad (16)$$

Although (15) is singular when $T_{r1}/\tau=2.11$ ($\zeta=\infty$), the inverse Laplace transform of the step response of a 1st order system (17) has the ratio $T_{r1}/\tau=2.2$. Also, the characteristic equation of a 2nd order system is used to show the high frequency pole p_2 is non-dominant ($p_2 \geq 10p_1$) when $\zeta>1.74$ (18) so (15) is not needed when $T_{r1}/\tau<2$ (see Table I) since a 1st order approximation is preferred.

$$\frac{T_{r1}}{\tau} = \ln(10) - \ln(90) = 2.2 \quad (17)$$

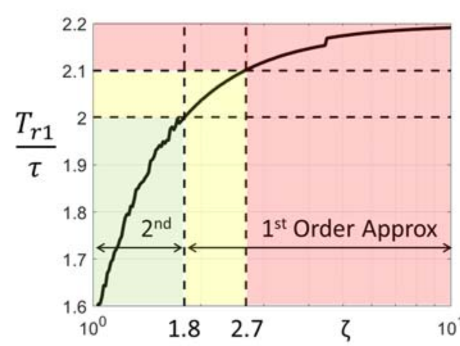
$$\zeta = \frac{p_1+p_2}{2\sqrt{p_1p_2}} = \frac{p_1+10p_1}{2p_1\sqrt{10}} = 1.74 \quad (18)$$

TABLE I. PREFERRED APPROXIMATION

Metric Ratio	*Zeta*	*Dominant Poles*	*Order of Approximation*
$T_{r1}/\tau < 2$	$1 < \zeta < 1.8$	2	2nd
$2 < T_{r1}/\tau < 2.1$	$1.8 < \zeta < 2.7$	1	1st preferred
$2.1 < T_{r1}/\tau$	$2.7 < \zeta$	1 pole only	1st

Fig. 6 experimentally shows the relationship between T_{r1}/τ and ζ for a 2nd order system where green indicates a 2nd order approximation, red indicates a 1st order approximation, and yellow may be either (1st order approximation preferred). Note that $T_{r1}/\tau \rightarrow 2.2$ as $\zeta \rightarrow \infty$, as predicted by (17).

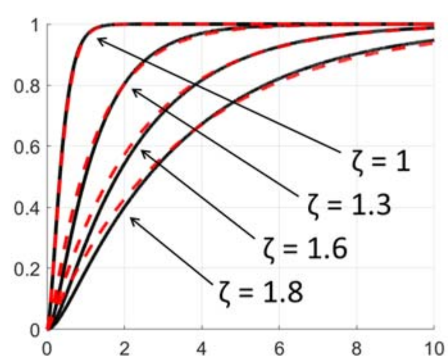
Figure 6. Time ratios vs Zeta.



Four over-damped step responses are approximated in Fig. 7. The one with a borderline non-dominant pole (ζ=1.8) uses a 1st order approximation (19), while the others use 2nd order approximations (15-16). The target curves are in black, the estimates are in red, and the correspondence of all four is excellent.

$$\frac{{1}/{\tau}}{(s+{1}/{\tau})} \quad (19)$$

Figure 7. Approximate 1st & 2nd Order Step Responses.



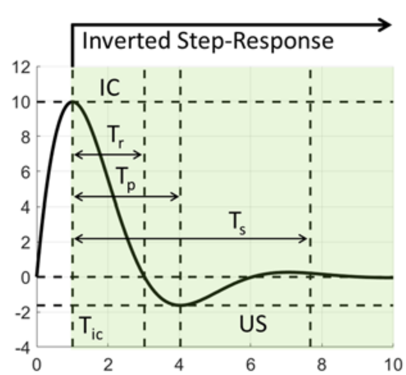
III. Impulse Response Identification

The impulse response of a 3rd order system with one pole at zero may be approximated using the same technique by simply treating it like a step response of a 2nd order system without a pole at zero (20).

$$1\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{1}{s}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad (20)$$

Almost the same is true for the impulse-response of a 2nd order system which contains all the same dynamic information as the step response, but has additional information that can be used to refine the approximation. The dynamics of a step response can be observed by raising a system to an initial condition and releasing it, which is essentially an impulse response. The time metrics T_r, T_p and T_s are all available by inspection, and so is a fourth time metric T_{ic}, the time required to raise the system to the initial condition, as shown in Fig. 8. Note that this argument does not hold for higher-order systems, since internal states may also have energy stored at T_{ic} so the "Inverted Step-Response" is not necessarily equivalent to the step-response when starting from rest.

Figure 8. Under-Damped 2nd Order Impulse Response Metrics.



When the impulse response of an under-damped system is available, T_r, T_p and T_s are available by inspection by simply measuring them relative to T_{ic}, with %OS derived from Percent Undershoot %US (21).

$$\%OS = \%US = \frac{-US}{IC} \quad (21)$$

To obtain T_{ic}, the impulse response of the generalized 2nd order equation (22) is transformed into the time domain (23) and T_{ic} is the minimum time t that sets the argument of the sine function in (23) equal to zero (24), providing a fourth redundant equation for ω_n. Unlike a step response, DC gain K_{dc} is not available by inspection, but substituting $t=T_{ic}$ into the impulse response (25) and equating it to IC, results in an equation for K_{dc} (26) from known values.

$$\frac{\omega_n^2 s}{s^2+2\zeta\omega_n s+\omega_n^2} \quad (22)$$

$$\frac{\sqrt{\omega_n}}{\beta}e^{-\zeta\omega_n t}\sin\left(\beta\omega_n t - \tan^{-1}\left(\beta/\zeta\right)\right) \quad (23)$$

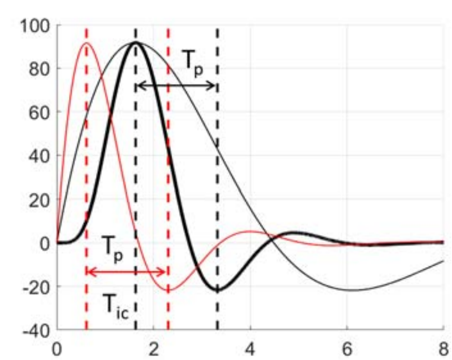
$$T_{ic} = \frac{\tan^{-1}\left(\beta/\zeta\right)}{\beta\omega_n} \qquad \omega_{nic} = \frac{\tan^{-1}\left(\beta/\zeta\right)}{\beta T_{ic}} \quad (24)$$

$$K_{dc}\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\xrightarrow{L^{-1}}K_{dc}\frac{\omega_n}{\beta}e^{-\zeta\omega_n t}\sin(\beta\omega_n t) \quad (25)$$

$$K_{dc} = \frac{IC}{\omega_n}e^{\zeta\omega_n T_{ic}} \quad (26)$$

In summary, a 2nd order under-damped impulse-response may be approximated using (3, 21) to compute ζ, then using (8-10, 24) to compute four redundant versions of ω_n which may either be selected individually or integrated in a weighted sum to arrive at an optimal ω_n, which is then combined with ζ and IC (26) to compute K_{dc}. For example, Fig. 9 shows two 2nd order approximations, the red one uses ω_{np} (9) to give an identical peak time T_p, while the black uses (24) to give an identical initial condition time T_{ic}. Note that the same system was used to generate the target curves in Fig. 2 and Fig. 9 but %OS in Fig. 2 does not equal %OS in Fig. 9 due to the higher-order (6th order in this example) target system so the particular %OS used to compute ζ is another free design parameter

Figure 9. Over-Damped 2nd Order Impulse Response Metrics.



A similar argument holds for over-damped systems. An over-damped impulse-response may be approximated using (15) to compute ζ, (16) to compute ω_n, and (26) to compute K_{dc} using the metrics shown in Fig. 10. Similarly, T_{ic} (24) may be used to more accurately estimate ω_n, since T_{ic} is exact, whereas (16) is derived from a linear approximation.

For over-damped systems, $\zeta>1$ so β is imaginary. However, since the inverse tangent ($\tan^{-1}$) of an imaginary number is also imaginary, both the numerator and denominator of (24) are imaginary and the result is both real and reliable. Critically-damped systems ($\zeta=1$) must be handled specially since (24) is singular and it may be necessary to artificially constrain ζ>=1 to prevent overshoot in the 2nd order approximation. The derivative of a critically-damped 2nd order system (27) is set to zero to compute T_{ic} (28), which is substituted into the time-domain response (29) to compute ω_{nic} (30) from values that are available by inspection of the step (K_{dc}) and impulse (IC) responses.

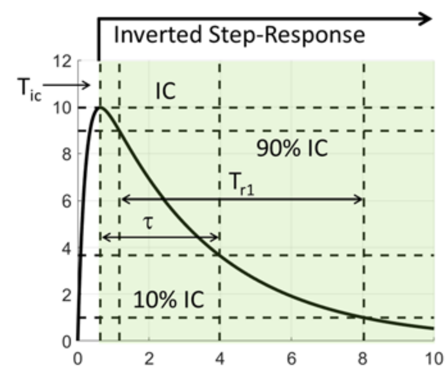
$$\frac{s{\omega_{nic}}^2}{(s+\omega_{nic})^2} \xrightarrow{L^{-1}} \omega_{nic}^2 e^{-\omega_{nic}t}(1-\omega_{nic}t) \quad (27)$$

$$\omega_{nic} = 1/T_{ic}; \quad \xi = 1 \quad (28)$$

$$K_{dc}\frac{{\omega_{nic}}^2}{(s+\omega_{nic})^2} \xrightarrow{L^{-1}} K_{dc}\omega_{nic}^2 t e^{-\omega_{nic}t} \quad (29)$$

$$IC = K_{dc}\omega_{nic}e^{-1} \quad \omega_{nic} = \frac{IC}{K_{dc}}e \quad (30)$$

Figure 10. Over-Damped 2nd Order Impulse Response Metrics.



IV. Example 2nd Order Approximations

When the target system is Black-Box, the approximation is made using the step response only. When the target system is White-Box, an initial approximation is made using the step response, and refined using the impulse response.

A. Black-Box Example

An electro-mechanical system (G_{sys}) system and sensor (H_{sen}) are supplied by a hypothetical company, "ACME Inc." who provides an experimental step response in each data sheet. The system G_{sys} has the under-damped step-response shown in Fig. 2, from which the metrics (31) are obtained. Physical units are not relevant for the sake of this example.

$$T_r = 2.1 \quad T_p = 2.7 \quad T_s = 4.9 \quad \%OS = 17 \quad K_{dc} = 100 \quad (31)$$

Equations (3, 8-10) are used to solve for the metrics (32).

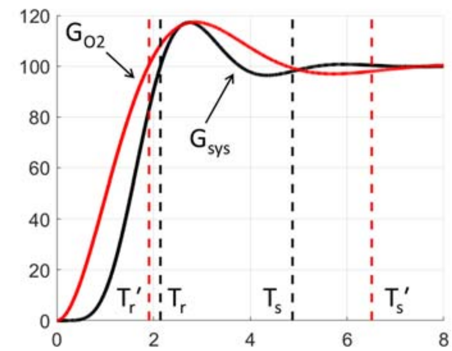
$$\zeta = 0.5 \quad \omega_{nr} = 1.1 \quad \omega_{np} = 1.3 \quad \omega_{ns} = 1.7 \quad (32)$$

The requirements, constraints and goals (RCGs) specify that rise time is three times as important as settle time, and peak time is relatively inconsequential. A weighted sum (33) is used to optimize ω_n and the associated 2nd order approximation G_{O2} (34) using ζ, ω_n and K_{dc} (31-32). The step response is shown in Fig. 11 where T_r' and T_s' are the rise and settle times of G_{O2}.

$$\omega_n = {(3\omega_{nr} + \omega_{ns})}/{4} = 1.25 \quad (33)$$

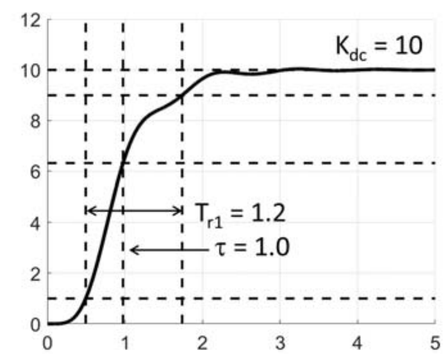
$$G_{O2} = \frac{158}{s^2+1.22s+1.58} \quad (34)$$

Figure 11. EM Step Response Approximation.



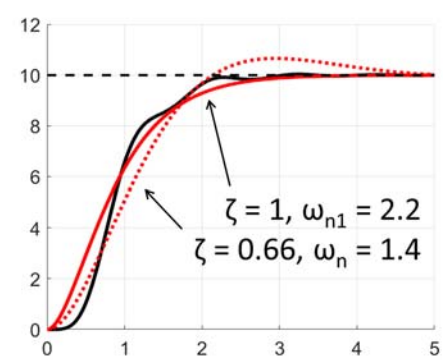
The sensor has the over-damped step-response shown in Fig. 12 with the metrics indicated.

Figure 12. Amplier Step Response.



Equations (15-16) are used to produce the approximations (ζ, ω_n) shown in Fig. 13, and to re-compute ω_{n1} with ζ=1, to satisfy the RCGs. Of course, ζ could have been constrained by a value other than 1 (e.g. ζ>=0.95) to satisfy a different overshoot constraint. Note that although (15) is designed for an apparently over-damped system, it returns ζ<1 if an under-damped approximation better satisfies the criteria.

Figure 13. Amplier Step Response Approximations.



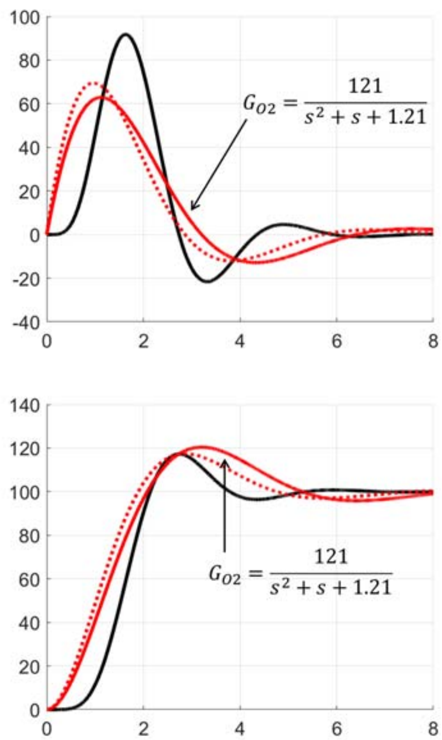
B. White-Box Example

ACME Inc. is internally developing a control system that incorporates its own system and sensor. For G_{sys}, the initial approximation from Fig 11 is first computed. Full information is available so impulse responses are produced, and used to refine the approximation using the average %OS from the step and impulse responses (35), using (3) to re-compute ζ, (24) to compute ω_{nic}, using the weighted sum (36) to place emphasis on T_{ic}, and with K_{dc} taken from the step response. The initial (dotted red) and refined (solid red) impulse & step-responses are shown in Fig. 14.

$$\%OS = 20 \quad \zeta = 0.45 \quad \omega_{nic} = 0.77 \quad \omega_{ns} = 2.1 \tag{35}$$

$$\omega_n = {(3\omega_{nic} + \omega_{ns})}/{4} = 1.1 \tag{36}$$

Figure 14. Tuned EM Response.

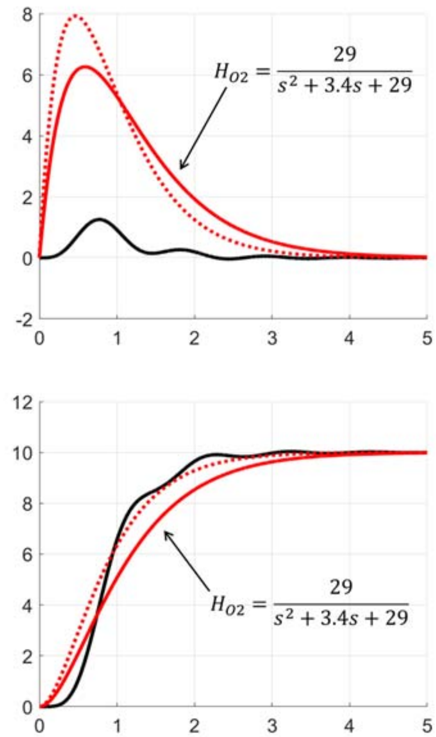


For G_{sys}, the initial approximation (ζ=1) from Fig 13 is first computed. Impulse responses are produced, and T_{ic} is used to compute ω_{nic} (28) which is averaged with ω_{n1} to refine ω_n. The initial (dotted red) and refined (solid red) impulse & step-responses are shown in Fig. 15.

$$T_{ic} = 0.77 \quad \omega_{nic} = 0.34 \tag{37}$$

$$\omega_n = {(\omega_{nic} + 3\omega_{n1})}/{4} = 1.7 \tag{38}$$

Figure 15. Tuned EM Response.



V. Conclusion

Developing a 2nd order approximation of a higher-order system is a subjective activity with an infinite number of solutions. The best solution provides an optimal compromise between metrics with different levels of importance, as determined by the application.

Here, closed-form equations are developed to extract all available 2nd order information from an over, under, or critically-damped impulse or step response. Redundant equations are combined in weighted sums to optimize the relative importance of reproducing different characteristics of the target response, resulting in a toolbox of equations that allows a design engineer to tune the approximation to meet a particular design specification without having to resort to any numeric or iterative computations.

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