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Boundary Tracking Control for An Unstable Wave Equation with Boundary Uncertainties: A Backstepping Adaptive NN Approach

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***Abstract*—This paper investigates the tracking control problem of an unstable wave equation with boundary uncertainties. The wave equation under consideration has a negative damper (unstable) at the uncontrolled boundary and uncertain nonlinear dynamics at the controlled boundary. A novel boundary tracking control scheme is proposed by incorporating the backstepping method with adaptive neural networks (NN). Specifically, an adaptive radial basis function (RBF) NN model is first developed to approximate/counteract the system uncertainties. A boundary-feedback observer is then designed with such a NN model to estimate the overall state of the wave equation. Based on this, a boundary tracking controller is finally proposed using the adaptive backstepping technique. Uniquely, this new control scheme is capable of rendering stable state tracking (i.e., driving the system's holistic state to track a prescribed reference trajectory), significantly advancing the current literature that is largely focused on output tracking control. Rigorous analysis is performed to verify the well-posedness and stability of the overall closed-loop system. Simulation studies have been conducted to demonstrate effectiveness of the proposed results.**

***Index Terms*—Distributed parameter systems, wave equation, boundary tracking control, backstepping, adaptive neural network.**

I. Introduction

STRING and flexible beams—usually modeled by wave/beam equations—are important benchmarks for the development of distributed parameter system theory [1]. They are crucial for many flexible distributed parameter systems, such as flexible manipulator [2] and flexible link robot arms [3]. Research on wave/beam equations has been attracting considerable attention over the past few decades, e.g., [1], [4], [5], [6].

Tracking control design of wave equations is an important problem from both theoretical and practical perspectives, owing to the ever-increasing demands of many applications, e.g., flexible robots in manufacturing [7], [8] which require the operating system's state/output to track a certain prescribed trajectory. In particular, boundary control design for such a tracking control problem of wave equations has been of interests, due to its practical advantages of demanding fewer sensors and actuators for controller implementations [6]. Many research efforts have been dedicated to this field. For example, researchers of [9] developed a boundary-feedback tracking control scheme for a wave equation with harmonic disturbances. [6] proposed an adaptive neural network (NN) based boundary control scheme for the reference-tracking problem of a wave equation with both matched and unmatched boundary uncertainties. [5] studied a wave equation with internal uncertainty and external disturbance. However, most of these schemes only considered the wave equations with dissipative/stable system operator (e.g., the systems of [9], [6], [5] have a positive damper at the uncontrolled boundary), whose associated open-loop systems are usually stable. For those unstable wave equations, e.g., the one in [10] that has a negative damper (unstable) at the uncontrolled boundary, the associated tracking control design is a rather challenging problem and still under-explored.

In this paper, we investigate the tracking control problem of an unstable uncertain wave equation, which has a negative damper at the uncontrolled boundary and uncertain nonlinear dynamics at the controlled boundary. A novel boundary adaptive tracking control scheme will be developed by: (i) utilizing the backstepping technique to handle the system's instability at the uncontrolled boundary; and (ii) employing adaptive NN technique to deal with the dynamic system uncertainties at the controlled boundary. More specific, we first develop an adaptive radial basis function neural network (RBF NN) model to approximate/counteract the system's uncertain dynamics. With this NN model, a boundary-feedback observer is then designed to estimate the overall system state of the wave equation. Using this observer, a boundary tracking control scheme is finally developed with the backstepping method and adaptive NN technique, which is capable of rendering stable and accurate tracking control for the wave equation. It is worth mentioning that our control scheme is able to drive the system's holistic state (instead of just system's output) to track a prescribed reference trajectory, which advances most of existing schemes, e.g., [9], [6], [5]. Rigorous analysis about the well-posedness and system stability of the overall closed-loop system is provided.

We would like to emphasize that the current research work significantly expands our previous work [6] by proposing a novel backstepping-adaptive-NN-based boundary tracking control scheme for a wave equation with boundary uncertainties. Specifically, different from [6] focusing on a stable wave equation, the current paper considers an unstable wave

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equation, whose tracking control design is more challenging. Moreover, distinguished from the scheme of [6] only using adaptive NN techniques, the approaches proposed in the current paper are developed by combining the backstepping method with adaptive NNs, which are capable of (i) guaranteeing the closed-loop stability for the unstable wave equation; and (ii) rendering accurate tracking for the system's holistic state.

The main contributions of this paper are summarized as follows. (i) The challenging problem of tracking control of unstable wave equations with boundary uncertainties is successfully addressed. (ii) The proposed boundary adaptive NN backstepping tracking control scheme is novel in the sense that it can provide stable holistic state tracking control (instead of only output tracking) for unstable wave equations. (iii) Rigorous analysis is performed to verify well-posedness and demonstrate stability of the overall closed-loop system.

The rest of this paper is organized as follows. Section II provides preliminary results and the problem formulation. Section III presents the design of a state-feedback backstepping control scheme. Section IV shows the proposed boundary-feedback adaptive NN backsetpping control scheme. Simulation results are presented in Section V. Conclusions are drawn in Section VI.

Notation. $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}_+$ denote, respectively, the set of real numbers, the set of positive real numbers and the set of positive integers; $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors; $|\cdot|$ is the absolute value of a real number; $\|\cdot\|$ is the 2-norm of a vector or a matrix; $L^\infty(\Omega)$ denotes the set of functions that are almost everywhere bounded on a measure space Ω; $(\cdot)_x$, $(\cdot)_{xx}$, $(\cdot)_t$, $(\cdot)_{tt}$ denote $\frac{\partial(\cdot)}{\partial x}$, $\frac{\partial^2(\cdot)}{\partial x^2}$, $\frac{\partial(\cdot)}{\partial t}$, $\frac{\partial^2(\cdot)}{\partial t^2}$, respectively; $\dot{(\cdot)}$ denotes $\frac{\partial(\cdot)}{\partial t}$.

II. Preliminaries and Problem Fomulation

A. Preliminaries

The RBF networks can be described by $f_{nn}(Z) = \sum_{i=1}^{N_n} w_i s_i(Z) = W^\top S(Z)$ [11], where $Z \in \Omega_Z \subset \mathbb{R}^q$ is the input vector, $W = [w_1, \cdots, w_{N_n}]^\top \in \mathbb{R}^{N_n}$ is the weight vector, N_n is the NN node number, and $S(Z) = [s_1(\|Z - \varsigma_1\|), \cdots, s_{N_n}(\|Z - \varsigma_{N_n}\|)]^\top$, with $s_i(\cdot)$ being a radial basis function, and ς_i $(i = 1, 2, \cdots, N_n)$ being distinct points in state space. The Gaussian function $s_i(\|Z - \varsigma_i\|) = \exp[\frac{-(Z-\varsigma_i)^\top(Z-\varsigma_i)}{\eta_i^2}]$ is one of the most commonly used radial basis functions, where $\varsigma_i = [\varsigma_{i1}, \varsigma_{i2}, \cdots, \varsigma_{iq}]^\top$ is the center of the receptive field and η_i is the width of the receptive field. The Gaussian function belongs to the class of localized RBFs in the sense that $s_i(\|Z - \varsigma_i\|) \to 0$ as $\|Z\| \to \infty$. According to [11], for any continuous function $f(Z) : \Omega_Z \to \mathbb{R}$ where $\Omega_Z \subset \mathbb{R}^q$ is a compact set, and for the NN approximator, where the node number N_n is sufficiently large, there exists an ideal constant weight vector W^*, such that for any $\epsilon^* > 0$, $f(Z) = W^{*\top} S(Z) + \epsilon$, $\forall Z \in \Omega_Z$, where $|\epsilon| < \epsilon^*$ is the ideal approximation error. The ideal weight vector W^* is an "artificial" quantity required for analysis, and is defined as the value of W that minimizes $|\epsilon|$ for all $Z \in \Omega_Z \subset \mathbb{R}^q$, i.e., $W^* \triangleq arg\min_{W \in \mathbb{R}^{N_n}} \{\sup_{Z \in \Omega_Z} |f(Z) - W^\top S(Z)|\}$.

B. Problem Formulation

Consider a one-dimensional unstable wave equation with boundary uncertainties:

$$\begin{cases} y_{tt}(x,t) = y_{xx}(x,t), \quad x \in (0,1),\, t \in (0,\infty) \\ y_x(0,t) = -q y_t(0,t) \\ y_x(1,t) = u(t) + f(y(1,t), y_t(1,t)) \\ y(x,0) = y_0(x),\, y_t(x,0) = y_1(x) \end{cases} \tag{1}$$

where $y(x,t) \in \mathbb{R}$ is the system state at the position $x \in [0,1]$ for time $t \in [0,\infty)$; $u \in \mathbb{R}$ is the system boundary control input; $q > 0$ $(q \neq 1)$ is a known constant; $f(y(1,t), y_t(1,t))$ is an unknown locally Lipschitz continuous nonlinear function, representing the system boundary uncertainty; $y_0(x)$ and $y_1(x)$ are initial conditions. Assume that the boundary signals $y(0,t)$, $y_t(0,t)$, $y(1,t)$, and $y_t(1,t)$ of (1) are all measurable.

In this paper, our objective is to design a boundary tracking control scheme for the system (1) with its boundary signals, i.e., $y(0,t)$, $y_t(0,t)$, $y(1,t)$ and $y_t(1,t)$, aiming to drive the system state $y(x,t)$ $(\forall x \in [0,1])$ to track a prescribed reference signal $r(x,t)$ with guaranteed well-posedness and system stability. In particular, note that the wave equation (1) has a negative damper at the uncontrolled boundary, i.e., $y_x(0,t) = -q y_t(0,t)$, which will lead the system to an unstable manner; furthermore, the system (1) has uncertain nonlinear dynamics $f(y(1,t), y_t(1,t))$, which will challenge the subsequent designs of system's state estimation and reference tracking. In view of this, our control scheme will be deigned by: (i) employing the backstepping technique to handle the system's instability at the boundary; and (ii) utilizing adaptive NN technique to deal with the system uncertainty $f(y(1,t), y_t(1,t))$.

Before proceeding, a reference model used to generate the desired reference signal $r(x,t)$ is constructed as follows:

$$\begin{cases} v_{tt}(x,t) = v_{xx}(x,t) \\ v_x(0,t) = c v_t(0,t) \\ v(1,t) = w_{ref}(t) \\ v(x,0) = v_0(x),\, v_t(x,0) = v_1(x), \end{cases} \tag{2}$$

where c is a design parameter satisfying $c > 0$, $c \neq 1$ and $qc \neq -1$; $w_{ref}(t)$ is a reference command that can be chosen freely to satisfy $w_{ref}(t) \in W^{2,\infty}(0,\infty) := \{w(t) \,|\, w \in L^\infty(0,\infty),\, w_t \in L^\infty(0,\infty),\, w_{tt} \in L^\infty(0,\infty)\}$; $v_0(x)$ and $v_1(x)$ are initial conditions. With this model, the generated reference signal $r(x,t)$ can be described by:

$$\begin{aligned} r(x,t) = &-\frac{1+qc}{c^2-1} v(x,t) + \frac{c(q+c)}{c^2-1} v(0,t) \\ &+ \frac{q+c}{c^2-1} \int_0^x v_t(\varsigma,t) d\varsigma. \end{aligned} \tag{3}$$

Lemma 1. *If the design parameters satisfy $c > 0$, $c \neq 1$, and $w_{ref}(t) \in W^{2,\infty}(0,\infty)$, the reference system (2)–(3) is well-posed and bounded.*

Proof. Well-posedness and system stability of the reference model (2) can be proved by following a similar line of our previous work [6, Lem. 2], which is omitted here. This can guarantee that the overall reference model (2)–(3) is well-posed and bounded. □

Remark 1. *The reference model (2)–(3) is designed based on a backstepping transformation, as will be discussed in (5)–(6). Such a setup can facilitate designing a backstepping controller for driving the system (1) to track the reference model (2)–(3) with guaranteed well-posedness and stability. We stress that implementation of (2)–(3) is feasible in practice. The signal $r(x,t)$ can be obtained with a suitable state $v(x,t)$, which can be generated by appropriately selecting the design parameter c and the reference command $w_{ref}(t)$ of (2). An example will be given in the simulation section for illustration.*

III. State-Feedback Backstepping Control

In this section, we will first present the design of a state-feedback backstepping control scheme for the system (1), which can enable the system state $y(x,t)$ to track the reference trajectory $r(x,t)$ of (3) with $\forall x \in [0,1]$. We assume that the system dynamics $f(y(1,t), y_t(1,t))$ in (1) is precisely known, and the system states $[y(x,t), y_t(x,t)]$ are measurable for all $x \in [0,1]$ and $t \in [0,\infty)$.

Specifically, for the system (1) and the reference model (2)–(3), a state-feedback backstepping controller is designed as follows:

$$\begin{aligned} u(t) = &-f(y(1,t), y_t(1,t)) - \frac{q^2-1}{1+qc}(c_0 v(1,t) + v_x(1,t)) \\ &- c_0 y(1,t) + \frac{q+c}{1+qc}(c_0 q y(0,t) - y_t(1,t) - c_0\int_0^1 y_t dx), \end{aligned} \tag{4}$$

where $c_0 > 0$ is a design parameter; q is the parameter from (1); and c is from (2).

Consider the system (1) with the controller (4), following a similar idea of [10], we define an invertible backstepping transformation:

$$\begin{aligned} w(x,t) = &-\frac{1+qc}{q^2-1}y(x,t) + \frac{q(q+c)}{q^2-1}y(0,t) \\ &- \frac{q+c}{q^2-1}\int_0^x y_t(\varsigma,t)d\varsigma. \end{aligned} \tag{5}$$

It can map the system (1) into the following system:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) \\ w_x(0,t) = c w_t(0,t) \\ w_x(1,t) = -c_0(w(1,t) - v(1,t)) + v_x(1,t) \\ y(x,t) = -\frac{1+qc}{c^2-1}w(x,t) + \frac{c(q+c)}{c^2-1}w(0,t) \\ \qquad + \frac{q+c}{c^2-1}\int_0^x w_t(\varsigma,t)d\varsigma. \end{cases} \tag{6}$$

By comparing the system (6) with the reference model (2)–(3), denoting $\varepsilon(x,t) = w(x,t) - v(x,t)$ and $e(x,t) = y(x,t) - r(x,t)$, we can obtain the following error dynamics:

$$\begin{cases} \varepsilon_{tt}(x,t) = \varepsilon_{xx}(x,t) \\ \varepsilon_x(0,t) = c\varepsilon_t(0,t) \\ \varepsilon_x(1,t) = -c_0\varepsilon(1,t) \\ e(x,t) = -\frac{1+qc}{c^2-1}\varepsilon(x,t) + \frac{c(q+c)}{c^2-1}\varepsilon(0,t) \\ \qquad + \frac{q+c}{c^2-1}\int_0^x \varepsilon_t(\varsigma,t)d\varsigma. \end{cases} \tag{7}$$

Theorem 1. *Consider the closed-loop system consisting of the plant (1), the reference model (2)–(3), and the controller (4). If the design parameters satisfy $c_0 > 0$, $c > 0$, $c \neq 1$ and $qc \neq -1$, we have: the closed-loop system is well-posed and exponentially stable in the sense of $(\int_0^1(e_x^2+e_t^2)dx + e^2(1,t))^{\frac{1}{2}}$ with $e(x,t) = y(x,t) - r(x,t)$.*

Proof. Following a similar line of the proof for [10, Th. 1], it can be proved that the error system (7) is well-posed and exponentially stable in the sense of $(\int_0^1(e_x^2 + e_t^2)dx + e^2(1,t))^{\frac{1}{2}}$. Then, noting that $y(x,t) = e(x,t) + r(x,t)$, and the reference model (2)–(3) is well-posed and bounded from Lemma 1, we have: the system (1) with controller (4) is well-posed and bounded. Consequently, it can be deduced that the closed-loop system of (1)–(4) is well-posed and bounded; moreover, it is exponentially stable in the sense of $(\int_0^1(e_x^2 + e_t^2)dx + e^2(1,t))^{\frac{1}{2}}$. This ends the proof. □

IV. Boundary-Feedback Adaptive NN Backstepping Control

In the previous section, the design of controller (4) requires the information of system dynamics $f(y(1,t), y_t(x,t))$ and the measurement of system's overall state $y_t(x,t)$, which could be very difficult in practice. To deal with these issues, in this section, we will design a boundary-feedback observer to estimate the overall state $[y(x,t), y_t(x,t)]$ of (1), in which adaptive NN technique will be used to deal with the effect of system uncertainty $f(y(1,t), y_t(1,t))$. With this observer, a more practical boundary-feedback backstepping controller can be designed.

A. Boundary-Feedback Adaptive NN Observer

For the system uncertain dynamics $f(y(1,t), y_t(1,t))$ in (1), according to Section II-A, we know that there exists a constant NN weight $W^* \in \mathbb{R}^{N_n}$ (with $N_n \in \mathbb{N}_+$ denoting the number of NN nodes) such that:

$$f(y(1,t), y_t(1,t)) = W^{*\top} S(y(1,t), y_t(1,t)) + \epsilon, \tag{8}$$

where $S(\cdot) : \mathbb{R}^2 \to \mathbb{R}^{N_n}$ is a smooth RBF vector, $\epsilon \in \mathbb{R}$ is the NN estimation error satisfying $|\epsilon| < \epsilon^*$ with ϵ^* being a small positive constant. Based on this, for the system (1), we can propose a boundary-feedback adaptive NN observer as follows:

$$\begin{cases} \hat{y}_{tt}(x,t) = \hat{y}_{xx}(x,t) \\ \hat{y}_x(0,t) = -q y_t(0,t) - c_1(y_t(0,t) - \hat{y}_t(0,t)) \\ \hat{y}_x(1,t) = u(t) + \hat{W}^\top S(y(1,t), y_t(1,t)) \\ \qquad + c_2(y(1,t) - \hat{y}(1,t)) \\ \hat{y}(x,0) = \hat{y}_0(x),\ \hat{y}_t(x,0) = \hat{y}_1(x) \\ \dot{\hat{W}} = \Gamma(\tilde{y}_t(1,t) + \delta_1 c_2 \tilde{y}(1,t))S - \Gamma\gamma\hat{W}, \end{cases} \tag{9}$$

where $y_t(0,t)$, $y(1,t)$ and $y_t(1,t)$ are system boundary signals of (1); q is the system parameter of (1); $\hat{W} \in \mathbb{R}^{N_n}$ is the estimate of W^* in (8); $c_1 > 0$, $c_2 > 0$, $\Gamma = \Gamma^\top > 0$, $\gamma > 0$ and $0 < \delta_1 < \min\{\frac{c_1}{1+c_1^2}, \frac{1}{2}\}$ are design parameters; $\tilde{y}(x,t) = y(x,t) - \hat{y}(x,t)$.

Comparing the observer (9) with the system (1), from (8), denoting $\tilde{W} = \hat{W} - W^*$, we obtain the error dynamics as:

$$\begin{cases} \tilde{y}_{tt}(x,t) = \tilde{y}_{xx}(x,t) \\ \tilde{y}_x(0,t) = c_1\tilde{y}_t(0,t) \\ \tilde{y}_x(1,t) = -c_2\tilde{y}(1,t) - \tilde{W}^\top S + \epsilon \\ \dot{\tilde{W}} = \Gamma(\tilde{y}_t(1,t) + \delta_1 c_2 \tilde{y}(1,t))S - \Gamma\gamma\hat{W}. \end{cases} \tag{10}$$

Lemma 2. *If the design parameters satisfy* $c_1 > 0$, $c_2 > 0$, $\Gamma = \Gamma^\top > 0$, $\gamma > 0$, *and* $0 < \delta_1 < \min\{\frac{c_1}{1+c_1^2}, \frac{1}{2}\}$, *the observation error dynamics system (10) is well-posed and bounded.*

Proof. Well-posedness of system (10) can easily be proved by following a similar line of the proof in our previous work [6, Th. 1], which thus is omitted here.

We study the stability of system (10). Define a Lyapunov function as: $V_1 = \frac{1}{2}\int_0^1(\tilde{y}_x^2 + \tilde{y}_t^2)dx + \frac{c_2}{2}\tilde{y}^2(1,t) + \delta_1\int_0^1(x-2)\tilde{y}_x\tilde{y}_t dx + \frac{1}{2}\tilde{W}^\top\Gamma^{-1}\tilde{W}$, which is positive definite since $0 < \delta_1 < \frac{1}{2}$. From (10), according to Young's inequality, the derivative of V_1 is derived as:

$$\begin{aligned} \dot{V}_1 = &-(c_1 - \delta_1(1+c_1^2))\tilde{y}_t^2(0,t) - \frac{\delta_1}{2}\tilde{y}_t^2(1,t) - \frac{\delta_1 c_2^2}{2}\tilde{y}^2(1,t) \\ &- \frac{\delta_1}{2}\int_0^1(\tilde{y}_x^2 + \tilde{y}_t^2)dx - \gamma\tilde{W}^\top\tilde{W} - \gamma\tilde{W}^\top W^* \\ &- \frac{\delta_1}{2}(\tilde{W}^\top S - \epsilon)^2 + \delta_1 c_2\tilde{y}(1,t)\epsilon + \tilde{y}_t(1,t)\epsilon \\ \le &-(c_1 - \delta_1(1+c_1^2))\tilde{y}_t^2(0,t) - \frac{\delta_1}{4}\tilde{y}_t^2(1,t) - \frac{\delta_1 c_2^2}{4}\tilde{y}^2(1,t) \\ &- \frac{\delta_1}{2}\int_0^1(\tilde{y}_x^2 + \tilde{y}_t^2)dx - \frac{\gamma}{2}\left\|\tilde{W}\right\|^2 + \frac{\gamma}{2}\|W^*\|^2 + \frac{\delta_1^2+1}{\delta_1}\epsilon^{*2} \end{aligned} \tag{11}$$

where $|\epsilon| < \epsilon^*$ from (8). Then, since $0 < \delta_1 < \frac{c_1}{1+c_1^2}$ leading to $c_1 - \delta_1(1+c_1^2) > 0$, we have $\dot{V}_1(t) < 0$ whenever:

$$\begin{aligned} \tilde{y}^2(1,t) \ge& \frac{4(\delta_1^2+1)}{\delta_1^2 c_2^2}\epsilon^{*2} + \frac{2\gamma}{\delta_1 c_2^2}\|W^*\|^2; \\ \int_0^1(\tilde{y}_x^2 + \tilde{y}_t^2)dx \ge& \frac{2(\delta_1^2+1)}{\delta_1^2}\epsilon^{*2} + \frac{\gamma}{\delta_1}\|W^*\|^2; \\ \left\|\tilde{W}\right\|^2 \ge& \frac{2(\delta_1^2+1)}{\gamma\delta_1}\epsilon^{*2} + \|W^*\|^2. \end{aligned} \tag{12}$$

This guarantees that the signals of $\tilde{y}(1,t)$, $\tilde{y}_x(x,t)$, $\tilde{y}_t(x,t)$ and $\tilde{W}$ of (10) are all bounded. Based on this and from the Poincare inequality, we have: all the signals of system (10), including $\tilde{y}(x,t)$, $\tilde{y}_t(x,t)$ and $\tilde{W}$ are bounded. This ends the proof. □

B. Boundary-Feedback Backstepping Controller

Using the observer (9), considering the system (1), from (4), we propose to design the following boundary-feedback adaptive NN backstepping controller:

$$\begin{aligned} u(t) =& -\hat{W}^\top S - \frac{q^2-1}{1+qc}(c_0 v(1,t) + v_x(1,t)) - c_0\hat{y}(1,t) \\ &+ \frac{q+c}{1+qc}(c_0 q\hat{y}(0,t) - \hat{y}_t(1,t) - c_0\int_0^1\hat{y}_t dx), \end{aligned} \tag{13}$$

where $\hat{y}(1,t)$, $\hat{y}_0(0,t)$ and $\hat{y}_t(x,t)$ are signals of observer (9); $\hat{W}^\top S$ is the adaptive NN model used in (9); $v(1,t)$, $v_x(1,t)$ are reference signals from (2).

Before investigating the performance of the controller (13) on the system (1), we first study its performance on the observer (9). We use the backstepping transformation of (5) to map the system (9) into the following system:

$$\begin{cases} \hat{w}_{tt}(x,t) = \hat{w}_{xx}(x,t) - \frac{(q+c)(c_1+q)}{q^2-1}\tilde{y}_{tt}(0,t) \\ \hat{w}_x(0,t) = c\hat{w}_t(0,t) + \frac{(1+qc)(c_1+q)}{q^2-1}\tilde{y}_t(0,t) \\ \hat{w}_x(1,t) = -c_0(\hat{w}(1,t) - v(1,t)) + v_x(1,t) \\ \qquad - \frac{(1+qc)c_2}{q^2-1}\tilde{y}(1,t). \end{cases} \tag{14}$$

Comparing this system with the reference model (2), and then combining with system (10), by denoting $\hat{\varepsilon}(x,t) = \hat{w}(x,t) - v(x,t)$, we obtain the overall error dynamics as follows:

$$\begin{cases} \hat{\varepsilon}_{tt}(x,t) = \hat{\varepsilon}_{xx}(x,t) - \frac{(q+c)(c_1+q)}{q^2-1}\tilde{y}_{tt}(0,t) \\ \hat{\varepsilon}_x(0,t) = c\hat{\varepsilon}_t(0,t) + \frac{(1+qc)(c_1+q)}{q^2-1}\tilde{y}_t(0,t) \\ \hat{\varepsilon}_x(1,t) = -c_0\hat{\varepsilon}(1,t) - \frac{(1+qc)c_2}{q^2-1}\tilde{y}(1,t) \\ \tilde{y}_{tt}(x,t) = \tilde{y}_{xx}(x,t) \\ \tilde{y}_x(0,t) = c_1\tilde{y}_t(0,t) \\ \tilde{y}_x(1,t) = -c_2\tilde{y}(1,t) - \tilde{W}^\top S + \epsilon \\ \dot{\tilde{W}} = \Gamma(\tilde{y}_t(1,t) + \delta_1 c_2\tilde{y}(1,t))S - \Gamma\gamma\hat{W}. \end{cases} \tag{15}$$

Lemma 3. *If the design parameters satisfy:* $c > 0$, $c \neq 1$, $qc \neq -1$, $c_0 > 0$, $c_1 > 0$, $c_2 > 0$, $\Gamma = \Gamma^\top > 0$, $\gamma > 0$, *and* $0 < \delta_1 < \min\{\frac{c_1}{1+c_1^2}, \frac{1}{2}\}$, *system (15) is well-posed and bounded.*

Proof. We consider a variable transformation: $\tilde{\varepsilon}(x,t) = \hat{\varepsilon}(x,t) + \frac{(q+c)(c_1+q)}{q^2-1}\tilde{y}(0,t)$, to rewrite system (15) as:

$$\begin{cases} \tilde{\varepsilon}_{tt}(x,t) = \tilde{\varepsilon}_{xx}(x,t) \\ \tilde{\varepsilon}_x(0,t) = c\tilde{\varepsilon}_t(0,t) - \frac{(c^2-1)(c_1+q)}{q^2-1}\tilde{y}_t(0,t) \\ \tilde{\varepsilon}_x(1,t) = -c_0\tilde{\varepsilon}(1,t) + \frac{(q+c)(c_1+q)c_0}{q^2-1}\tilde{y}(0,t) \\ \qquad - \frac{(1+qc)c_2}{q^2-1}\tilde{y}(1,t) \\ \tilde{y}_{tt}(x,t) = \tilde{y}_{xx}(x,t) \\ \tilde{y}_x(0,t) = c_1\tilde{y}_t(0,t) \\ \tilde{y}_x(1,t) = -c_2\tilde{y}(1,t) - \tilde{W}^\top S + \epsilon \\ \dot{\tilde{W}} = \Gamma(\tilde{y}_t(1,t) + \delta_1 c_2\tilde{y}(1,t))S - \Gamma\gamma\hat{W}. \end{cases} \tag{16}$$

It is easily seen that well-posedness and stability of the original system (15) can be investigated by studying the system (16).

Specifically, the well-posedness of system (16) can be proved by following a similar line of our previous work [6, Th. 1], which is omitted here. We investigate its system stability in the following. Define a positive-definite Lyapunov function: $V = KV_1 + V_2$, where $K > 0$, $V_1 = \frac{1}{2}\int_0^1(\tilde{y}_x^2 + \tilde{y}_t^2)dx + \frac{c_2}{2}\tilde{y}^2(1,t) + \delta_1\int_0^1(x-2)\tilde{y}_x\tilde{y}_t dx + \frac{1}{2}\tilde{W}^\top\Gamma^{-1}\tilde{W}$ as given in (11), and V_2 is defined as: $V_2 = \frac{1}{2}\int_0^1(\tilde{\varepsilon}_x^2 + \tilde{\varepsilon}_t^2)dx + \frac{c_0}{2}\tilde{\varepsilon}^2(1,t) + \delta_2\int_0^1(x-2)\tilde{\varepsilon}_x\tilde{\varepsilon}_t dx$, with $0 < \delta_2 < \min\{\frac{c}{2+4c^2}, \frac{1}{2}\}$.

From (16), according to Young's inequality and following a similar line of (11), the derivative of V_2 is obtained as:

$$\dot{V}_2 \le -(\frac{c}{2} - \delta_2(1+2c^2))\tilde{\varepsilon}_t^2(0,t) - \frac{\delta_2}{4}\tilde{\varepsilon}_t^2(1,t) - \frac{\delta_2 c_0^2}{4}\tilde{\varepsilon}^2(1,t)$$

$$
\begin{aligned}
&-\frac{\delta_2}{2}\int_0^1(\tilde{\varepsilon}_x^2+\tilde{\varepsilon}_t^2)dx+(\frac{2}{\delta_2}+2\delta_2)(\frac{(1+qc)c_2}{q^2-1})^2\tilde{y}^2(1,t)\\
&+(\frac{1}{2c}+2\delta_2)(\frac{(c^2-1)(c_1+q)}{q^2-1})^2\tilde{y}_t^2(0,t)\\
&+(\frac{2}{\delta_2}+2\delta_2)(\frac{(q+c)(c_1+q)c_0}{q^2-1})^2\tilde{y}^2(0,t).
\end{aligned}
\tag{17}
$$

From the Cauchy-Schwarz inequality and Young's inequality, we have: $-\int_0^1\tilde{y}_x^2dx\leq-(\int_0^1\tilde{y}_xdx)^2\leq-\frac{1}{2}\tilde{y}^2(0,t)+\tilde{y}^2(1,t)$, leading to $\tilde{y}^2(0,t)\leq 2\int_0^1\tilde{y}_x^2dx+2\tilde{y}^2(1,t)$. Based on this, combining (IV-B) with (11), the derivative of $V=KV_1+V_2$ is obtained as:

$$
\begin{aligned}
\dot{V}\leq&-(\frac{c}{2}-\delta_2(1+2c^2))\tilde{\varepsilon}_t^2(0,t)-\frac{\delta_2}{4}\tilde{\varepsilon}_t^2(1,t)-\frac{\delta_2c_0^2}{4}\tilde{\varepsilon}^2(1,t)\\
&-\frac{\delta_2}{2}\int_0^1(\tilde{\varepsilon}_x^2+\tilde{\varepsilon}_t^2)dx-(K(c_1-\delta_1(1+c_1^2))\\
&-(\frac{1}{2c}+2\delta_2)(\frac{(c^2-1)(c_1+q)}{q^2-1})^2)\tilde{y}_t^2(0,t)-K\frac{\delta_1}{4}\tilde{y}_t^2(1,t)\\
&-(K\frac{\delta_1c_2^2}{4}-(\frac{2}{\delta_2}+2\delta_2)(\frac{(1+qc)c_2}{q^2-1})^2-(\frac{4}{\delta_2}+4\delta_2)\\
&\cdot(\frac{(q+c)(c_1+q)c_0}{q^2-1})^2)\tilde{y}^2(1,t)-K\frac{\delta_1}{2}\int_0^1\tilde{y}_t^2dx\\
&-(K\frac{\delta_1}{2}-(\frac{4}{\delta_2}+4\delta_2)(\frac{(q+c)(c_1+q)c_0}{q^2-1})^2)\int_0^1\tilde{y}_x^2dx\\
&-K\frac{\gamma}{2}\left\|\tilde{W}\right\|^2+K\frac{\gamma}{2}\left\|W^*\right\|^2+K\frac{\delta_1^2+1}{\delta_1}\epsilon^{*2}.
\end{aligned}
\tag{18}
$$

Based on this, with $0<\delta_2<\frac{c}{2+4c^2}$, $0<\delta_1<\frac{c_1}{1+c_1^2}$, $\gamma>0$ and a sufficiently large value of $K>0$, following a similar line of the proof in Lemma 2, we have: all signals of the error system (16), including $\tilde{\varepsilon}(x,t)$, $\tilde{\varepsilon}_t(x,t)$, $\tilde{y}(x,t)$, $\tilde{y}_t(x,t)$ and $\tilde{W}$, are bounded. Consequently, we can conclude that the original error dynamics (15) is well-posed and bounded. This ends the proof. □

With Lemmas 1–3, we can establish the well-posedness and stability of the overall closed-loop system under our approaches as follows. In particular, from (1), (2), (3), (5), (7), (9) and (13), we obtain the overall error dynamic system as follows:

$$
\begin{cases}
\varepsilon_{tt}(x,t)=\varepsilon_{xx}(x,t)\\
\varepsilon_x(0,t)=c\varepsilon_t(0,t)\\
\varepsilon_x(1,t)=-c_0\varepsilon(1,t)+\frac{1+qc}{q^2-1}(\tilde{W}^\top S-\epsilon)-\frac{1+qc}{q^2-1}c_0\tilde{y}(1,t)\\
\qquad\qquad+\frac{q+c}{q^2-1}(c_0q\tilde{y}(0,t)-\tilde{y}_t(1,t)-c_0\int_0^1\tilde{y}_tdx)\\
e(x,t)=-\frac{1+qc}{c^2-1}\varepsilon(x,t)+\frac{c(q+c)}{c^2-1}\varepsilon(0,t)\\
\qquad\qquad+\frac{q+c}{c^2-1}\int_0^x\varepsilon_t(\varsigma,t)d\varsigma\\
\tilde{y}_{tt}(x,t)=\tilde{y}_{xx}(x,t)\\
\tilde{y}_x(0,t)=c_1\tilde{y}_t(0,t)\\
\tilde{y}_x(1,t)=-c_2\tilde{y}(1,t)-\tilde{W}^\top S+\epsilon\\
\dot{\tilde{W}}=\Gamma(\tilde{y}_t(1,t)+\delta_1c_2\tilde{y}(1,t))S-\Gamma\gamma\hat{W}.
\end{cases}
\tag{19}
$$

Theorem 2. *Consider the closed-loop system consisting of the plant (1), the reference model (2)–(3), the observer (9), and the controller (13). If the design parameters satisfy:* $c>0$, $c\neq 1$, $qc\neq-1$, $c_0>0$, $c_1>0$, $c_2>0$, $\Gamma=\Gamma^\top>0$, $\gamma>0$, *and* $0<\delta_1<\min\{\frac{c_1}{1+c_1^2},\frac{1}{2}\}$, *the closed-loop system is well-posed and bounded.*

Proof. Consider the error system (19). From (15), we have: $\varepsilon(x,t)=\hat{\varepsilon}(x,t)-\frac{1+qc}{q^2-1}\tilde{y}(x,t)+\frac{q(q+c)}{q^2-1}\tilde{y}(0,t)-\frac{q+c}{q^2-1}\int_0^x\tilde{y}_t(\varsigma,t)d\varsigma$. Since the system (15) is well-posed and bounded from Lemma 3, it is easy to deduce that the system (19) is well-posed and bounded. Based on this, noting that $y(x,t)=e(x,t)+r(x,t)$, $\hat{y}(x,t)=y(x,t)-\tilde{y}(x,t)$, and the reference model (2)–(3) is well-posed and bounded from Lemma 1, we have: both the system (1) and the observer (9) under the controller (13) are well-posed and bounded. Consequently, it is verified that the closed-loop system consisting of (1), (2), (3), (9) and (13) are well-posed and bounded. □

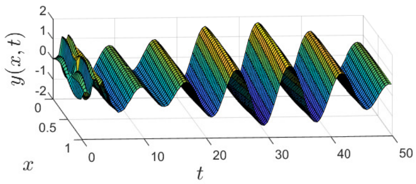
V. Simulation Studies

To demonstrate the effectiveness of our approaches, this section will perform a simulation study by using a numerical example. Specifically, the system (1) is given with $q=0.6$, $f(y(1,t),y_t(1,t))=0.4y(1,t)+0.3\cos(y_t^2(1,t))$, $y_0(x)=0.3\sin(2\pi x)$ and $y_1(x)=0.3\cos(2\pi x)$. The reference model (2)–(3) is designed with $c=0.7$, $w_{ref}(t)=0.5\sin(\frac{\pi}{5}t)+\cos(\frac{\pi}{4}t)$, $v_0(x)=0.1$ and $v_1(x)=0.1$. The observer (9) is designed with $c_1=0.7$, $c_2=1.2$, $\delta_1=0.45$, $\Gamma=3$, $\gamma=0.01$, $\hat{y}_0(x)=0.1$, $\hat{y}_1(x)=0.1$, $\hat{W}(0)=0$, and the RBF NN model $\hat{W}^\top S$ is constructed in a regular lattice with the number of nodes $N_n=5\times 13$, the centers evenly spaced on $[-2,2]\times[-7,5]$, and the widths $\eta_i=1$ $(i=1,2,\cdots,65)$. The controller (13) is with $c_0=1.2$.

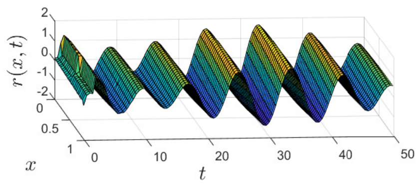
Performance of the overall closed-loop system consisting of the plant (1), the reference model (2)–(3), the observer (9) and the controller (13) are shown in Figs. 1–2. It is shown that all signals of the closed-loop system, including system state $y(x,t)$, reference signal $r(x,t)$, observer state $\hat{y}(x,t)$, control signal $u(t)$ and NN weight $\hat{W}(t)$, are stable. The signal-tracking performances of our scheme are illustrated in Fig. 3, showing that both the state estimation error $y(x,t)-\hat{y}(x,t)$ of (1) and (9), and the state tracking error $y(x,t)-r(x,t)$ of (1) and (3), can converge to a small neighborhood around zero. Consequently, these simulation results verify that our approaches can provide desired performance of closed-loop stability, state estimation, as well as reference tracking control. In particular, the results of Fig. 3 also justify that our control scheme can drive the system's holistic state $y(x,t)$ to track the reference signal $r(x,t)$ with $\forall x\in[0,1]$, which is advanced over most of the existing schemes of [9], [6], [5] that only achieve output tracking.

VI. Conclusions

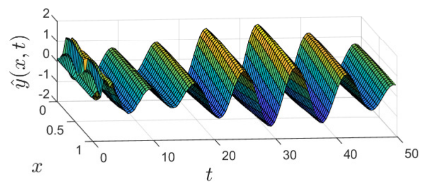
In this paper, we have developed a backstepping adaptive NN boundary control scheme for the tracking control problem of an unstable wave equation with boundary uncertainties.



(a) System state $y(x,t)$ of (1).

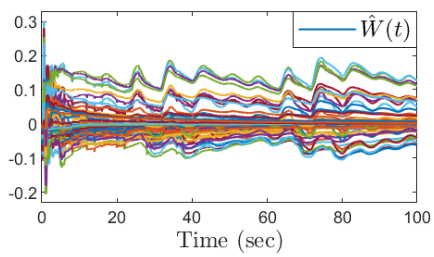


(b) Reference signal $r(x,t)$ of (3).

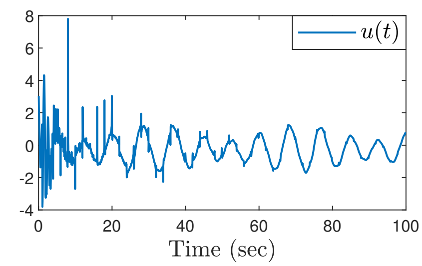


(c) Observer state $\hat{y}(x,t)$ of (9).

Fig. 1: Overall system state.

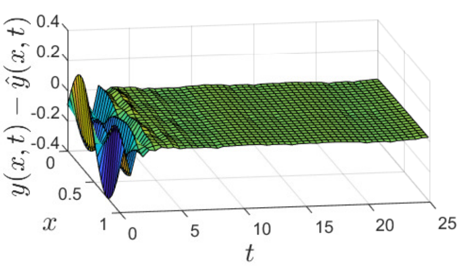


(a) NN weights $\hat{W}(t)$ of (9).

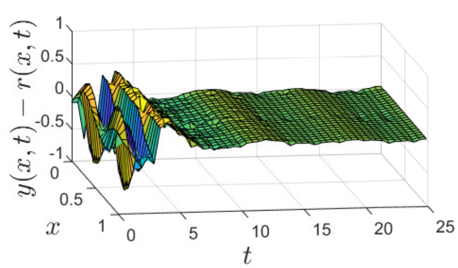


(b) Control signal $u(t)$ of (13).

Fig. 2: Closed-loop system response.



(a) State estimation error $y(x,t) - \hat{y}(x,t)$ of (1) and (9).



(b) State tracking error $y(x,t) - r(x,t)$ of (1) and (3).

Fig. 3: Signal-tracking performance.

Specifically, an adaptive RBF NN model has been developed to approximate/counteract the effect of system uncertain dynamics. A boundary-feedback observer has been designed with the NN model, to estimate the system's overall state. Using this observer, a boundary tracking control scheme has been proposed based on the backstepping method and adaptive NN technique. This control scheme is able to provide stable tracking control for the unstable wave equation, in which the system's holistic state can be driven to track a prescribed reference trajectory. Rigorous analysis has been performed to demonstrate the well-posedness and stability of the overall closed-loop system. In the future work, we expect to extend the proposed tracking control scheme to a more general case, e.g., an unstable wave equation with both matched and unmatched boundary uncertainties.

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