Distributed Control of Flexible Payload Transportation Using Multiple Quadrotors

Ti Chen
College of Aerospace Engineering
Nanjing University of Aeronautics and Astronautics
Nanjing, China
chent@nuaa.edu.cn

Jinjun Shan
Lassonde School of Engineering
York University
Toronto, Canada
jjshan@yorku.ca

Hugh H.T. Liu
Institute for Aerospace Studies
University of Toronto
Toronto, Canada
liu@utias.utoronto.ca

Abstract—This paper focuses on the distributed control of flexible payload transportation using multiple quadrotors. To design a distributed controller, the transportation system is described as a group of quadrotors subject to disturbance forces and torques from the flexible payload. A distributed finite-time observer is introduced for the quadrotors to estimate the virtual leader’s information. The disturbances acting on quadrotors due to the transverse and torsional deformation of the flexible payload are estimated based on the deformation analysis. The unmodeled disturbances are approximated using a radial basis function neural network. A distributed hierarchical controller is developed with the compensation for these disturbances.

Index Terms—Distributed control, Aerial transportation, Flexible payload, Quadrotor

I. INTRODUCTION

In the past decade, with the development of high-performance sensors and micro-scale processors, more and more quadrotors have been used to grasp, manipulate and transport objects [1]. The payload attached to the quadrotors will change the dynamic behaviors of the quadrotors. Consequently, aerial payload transportation opens some challenges in dynamics analysis, state estimation, path planning and controller design. Since the payload capability of a single quadrotor is limited, multiple quadrotors can be used to transport heavy payloads. Generally, cables or some rigid connectors, such as manipulators, grippers and magnets, can be used to attach the payload on the quadrotors. The cable-suspended payload will result in a complicated dynamic behavior of the transportation system especially considering the cable flexibility or the possible cable slackness. Hence, some studies have focused on such a challenging topic [2]–[5].

To enable the direct control of the payload, some works have studied the cooperative transportation of the payload attached on the quadrotor rigidly. Umemoto et al. studied the object transportation via the friction forces between the object and quadrotors based on a decentralized controller [6]. Mellinger et al. rigidly connected a payload with multiple quadrotors with all propellers in the parallel plane, modeled the entire system as a large multi-rotor Unmanned Aerial Vehicles (UAV) and proposed a hierarchical controller and control command allocation method based on the relative pose between quadrotors [7]. A similar dynamics and control framework is adopted in [8] for quadrotors carrying a payload via permanent magnets based on inertial sensing and monocular vision. To reduce the system complexity, the rigid connections via some simple grippers and magnets will be considered in this study.

When transporting flexible payloads, such as the fabrics in [9], flexible hose in [10] and flexible ring in [11], the quadrotor movement may result in the payload vibration, which will cause the time-varying relative orientation between the quadrotors in the team. Hence, the control method for the transportation of a rigid payload in [7], [8] does not work for transporting the flexible payload. Kotaru and Sreenath constructed a coordinate-free dynamic model by treating the flexible hose as a series of small links, analyzed the differential flatness of the entire system and designed a linear-quadratic regulator (LQR) based on the linearized dynamic model for the cooperative transportation of a flexible hose [10]. To reduce the computational and communication burden of the leader, a decentralized or distributed controller is preferred. This work focuses on the transportation of flexible beam using multiple quadrotors and the distributed control of flexible beam transportation using multiple quadrotors has not been reported in the literature. Essentially, the transportation task of a flexible beam can be considered as the quadrotor formation flying problem subject to some special disturbances from flexible deformations. During the past decade, the formation flying of multiple quadrotors has been well studied [12]. However, different from the classical quadrotor formation flying control, the disturbances acting on the quadrotors are mainly from the flexible deformation for the cooperative transportation in this study. Compared with the distributed control of multiple quadrotors in [12], [13], the beam deformation is analyzed based on the states of the quadrotor and its neighbors to estimate partial disturbances acting on the quadrotors.

II. SYSTEM MODEL

A. Notations and Graph Theory

\[ I_n \in \mathbb{R}^{n \times n} \] and \( 0_{m \times n} \in \mathbb{R}^{m \times n} \) represent the identity and zero matrices, respectively. \( \text{sign}(A) \) is defined as \( [\text{sign}(a_{ij})] \) with \( 1 > \alpha > 0, A = [a_{ij}] \) and \( \text{sign}(a_{ij}) = \text{sgn}(a_{ij})/|a_{ij}|^\alpha \), where \( \text{sgn}(\cdot) \) is the standard sign function. \( \text{sign}(A) \) represents the matrix \( [\text{sgn}(a_{ij})] \).

\[ SO(3) = \{ R \in \mathbb{R}^{3 \times 3} | R^T R = RR^T = I_3, \det(R) = 1 \} \]
is the set of $3 \times 3$ orthogonal matrices with determinant of one. $\text{so}(3) = \{\Omega \in \mathbb{R}^{3 \times 3} | \Omega = -\Omega^T \}$ denotes the Lie algebra of $SO(3)$. For any vector $\omega = [\omega_1, \omega_2, \omega_3]^T$, $\omega^\times$ is defined as the skew-symmetric matrix

$$
\begin{bmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{bmatrix}
$$

the mapping such that $(\omega^\times)^\times = \omega$. Based on the standard inner product on $\mathbb{R}^3$, an inner product on $\text{so}(3)$ is defined as $\langle \omega^1, \omega^2 \rangle = \frac{1}{2} \text{tr}(\omega^1 \omega^2) = \langle \omega_1, \omega_2 \rangle = \omega_1^2 + \omega_2^2$, where $\omega_1 \in \mathbb{R}^3$ and $\omega_2 \in \mathbb{R}^3$. For any angle $\theta$, $c_\theta$ and $s_\theta$ represent $\cos \theta$ and $\sin \theta$, respectively. $\otimes$ represents the Kronecker product defined as $A \otimes B = \begin{bmatrix}
\bar{a}_{i1}B & \cdots & \bar{a}_{i1}B \\
\vdots & \ddots & \vdots \\
\bar{a}_{i1}B & \cdots & \bar{a}_{i1}B
\end{bmatrix}$ for $A = [\bar{a}_{ij}] \in \mathbb{R}^m \times n$ and $B \in \mathbb{R}^q \times q$.

In this study, the communication among followers can be described by an undirected graph $\mathcal{G} = \{V, E\}$. $V = \{1, 2, \cdots, n\}$ and $E$ are the node and edge sets, respectively. The element $(i, j) \in E$ means that the $j$th node has the access to the information of node $i$. If $(i, j) \in E$ implies $(j, i) \in E$ for any edge, the graph is said to be undirected. A path from node $i_1$ to node $i_m$ is a sequence of nodes $i_1, i_2, \cdots, i_m$ such that $(i_{m-1}, i_m) \in E$ for $m = 1, 2, \cdots, m - 1$. The graph is said to be connected if there is a path between any two nodes. The adjacency matrix $A$ is defined as $[a_{ij}]$, where $a_{ij} = 1$ if $(j, i) \in E$ and $a_{ij} = 0$ otherwise. No self-loop is considered here, hence, $a_{ii} = 0$. The graph Laplacian matrix $L = D - A$, where $D = \text{diag}(d_i)$ with $d_i = \sum_{j=1}^n a_{ij}$. Suppose that the desired state is determined by a virtual leader. If the $i$th node can receive the leader’s information, $b_i = 1$ holds; otherwise, $b_i = 0$. Denote $B = \text{diag}\{b_i\}$. $H$ is defined as $L + B$.

### B. Quadrotor Dynamic Model

As shown in Fig. 1, $n$ identical quadrotors are attached to the flexible beam rigidly. The inertial frame $OXYZ$ is built to describe the system movement. The dynamic model of the $i$th quadrotor in the transportation system can be written as $m_q \ddot{z}_i = -m_q \ddot{z}_i + f_i R_i \dot{z}_b + d_{p,i}$, $\dot{R}_i = R_i \Omega_i \times$ and $J_i \ddot{\Omega}_i = -\Omega_i \times J_i \Omega_i + \tau_i + d_{\Omega,i}$, where $m_q$, $J_q = \text{diag}\{J_{q1}, J_{q2}, J_{q3}\}$, $\dot{r}_{qi} = [X_i, Y_i, Z_i]^T$ and $\Omega_i$ are the mass, inertia matrix, position vector and the angular velocity of the quadrotor. $g = 9.81 \text{ m/s}^2$ is the constant gravitational acceleration. $z_b = [0, 0, 1]^T$, $\tau_i = [\tau_{i1}, \tau_{i2}, \tau_{i3}]^T$ are the lift force and control torque vector generated by four propellers. $d_{p,i}$ and $d_{\Omega,i}$ are the disturbance forces and torques acting on the $i$th quadrotor from the flexible payload. $R_i \in SO(3)$ represents the rotation matrix from the body frame of the $i$th quadrotor to the frame $xyz$. Based on the 3-2-1 Euler angles, $R_i$ can be written as $R_i = 

\begin{bmatrix}
\cos \phi_i \cos \psi_i & \cos \phi_i \sin \psi_i & -\sin \phi_i \\
-\sin \theta_i \sin \phi_i \cos \psi_i + \cos \theta_i \sin \psi_i & -\sin \theta_i \sin \phi_i \sin \psi_i + \cos \theta_i \cos \psi_i & \cos \theta_i \sin \phi_i \\
\sin \theta_i \cos \phi_i \cos \psi_i + \cos \theta_i \sin \psi_i & \sin \theta_i \cos \phi_i \sin \psi_i + \cos \theta_i \cos \psi_i & -\sin \theta_i \sin \phi_i
\end{bmatrix}
$

where $\phi_i$, $\theta_i$, and $\psi_i$ are the roll, pitch and yaw angles of the $i$th quadrotor, respectively. With the assumptions of small roll and pitch angles, the attitude dynamics can be written as

$$
J_i \ddot{\psi}_i = -\dot{\psi}_i \times (J_q \dot{\psi}_i) + \tau_i + d_{a,i}
$$

(1)

where $\Psi_i = [\dot{\phi}_i, \dot{\theta}_i, \dot{\psi}_i]^T$.

### C. Dynamic Model of the Transportation System

This section aims to provide a dynamic model of such a multibody system based on the assumed modes method. This model will be used as the control plant in numerical simulations. Suppose that the flexible beam is attached almost at the mass centers of quadrotors. The length of each part of the beam is $l$. As shown in Fig. 1, $n$ body frames are built. In the inertial frame, $r_{qi} = [X_i, Y_i, Z_i]^T$ is used to describe the mass center of the $i$th quadrotor, where $i = 1, \cdots, n$. Hence, the position vector of an arbitrary point $P$ on the $i$th part of the beam is $r_{pi} = r_{qi} + R_i R_{iy}(y, t) [0, y, w_i(y, t)]^T$, where $i = 1, \cdots, n - 1$, $R_i \in SO(3)$ is the rotation matrix from the frame $x_i y_i z_i$ to the frame $OXYZ$, $y \in [0, l]$ represents the lengthwise position of this point in the frame $x_i y_i z_i$, and $w_i(y, t)$ is the lateral deformation. $R_{iy}(y, t) = \begin{bmatrix}
\cos \Theta_i(y, t) & 0 & \sin \Theta_i(y, t) \\
0 & 1 & 0 \\
-\sin \Theta_i(y, t) & 0 & \cos \Theta_i(y, t)
\end{bmatrix}$ is the rotation matrix representing the attitude change due to the torsion at point $P$, where $\Theta_i(y, t)$ is the torsional angle. The position vector of the $(i + 1)$th quadrotor can be written as $r_{qi+1} = r_{qi} + R_i R_{iy}(y, t) [0, l, w_i(y, t)]^T$. Considering both the lateral and torsional deformations, the relationship between $R_i$ and $R_{i+1}$ is $R_{i+1} = R_i R_{iy}(y, t)$, where $R_{iy} = \begin{bmatrix}
0 & 0 & 0 \\
\cos \varphi_i & 0 & \sin \varphi_i \\
-\sin \varphi_i & 0 & \cos \varphi_i
\end{bmatrix}$.

The angular velocity of the quadrotor $i + 1$ can be written as $\dot{\Omega}_{i+1} = (\dot{R}_{i+1}^T, \dot{R}_{i+1})^T$. Therefore, the angular velocity of each quadrotor can be expressed by the Euler angles of the first quadrotor, the deformations of all parts of the flexible beam, and their time derivatives.

Hence, the total kinetic and potential energies of the transportation system are $T = \frac{1}{2} \sum_{i=1}^n m_q (\dot{r}_{qi})^2 + \frac{1}{2} \sum_{i=1}^n m_q (\dot{r}_{qi})^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{n-1} \rho \langle \dot{r}_{qi}, \dot{r}_{qj} \rangle dy + U = \sum_{i=1}^n m_q g (r_{qi}, z_i) + \sum_{i=1}^n \int_0^l \rho g (r_{qi}, z_i) dy + \frac{1}{2} EI \sum_{i=1}^{n-1} \int_0^l (\dot{\varphi}^2 + \dot{\psi}^2) dy + \frac{1}{2} GJ_p \sum_{i=1}^n \int_0^l \left( \frac{\partial^2 w_i(y, t)}{\partial y^2} \right)^2 dy$, where $\rho$, $EI$ and $GJ_p$ are the linear density, the bending stiffness and torsional rigidity of the flexible beam. The assumed modes method can be used to
discretize the flexible beam, i.e., \( w_i(y, t) = \sum_{k=1}^{n_1} \Phi_k(y) p_k(t) \) and \( \Theta_i(y, t) = \sum_{k=1}^{n_2} \chi_k(y) q_k(t) \) if the first \( n_1 \) and \( n_2 \) modes are adopted for the bending and torsional deformation, and \( p_k(t) \) and \( q_k(t) \) are the \( k \)th order modal coordinates.

With \( q = \{X_1, Y_1, Z_1, \phi_1, \theta_1, p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2}\}^T \in \mathbb{R}^{6+n_1+n_2} \) as the generalized coordinate vector, based on the Lagrange’s equations of the second kind, the dynamic equation of the transportation system can be expressed as \( M\ddot{q} + C\dot{q} + Kq + g = Q \), where \( M, C, K \) are the generalized inertia matrix, the Coriolis and centrifugal matrix and the stiffness matrix, respectively. \( g \) is the gravity term. \( Q \) is the generalized force vector, which can be calculated based on the virtual work \( \delta W_v = \sum_{i=1}^{n} \langle \delta r_{ai}, f_i, R_e e_i \rangle + \sum_{i=1}^{n} \langle R_i^T \delta R_i, \tau_i^T \rangle = Q^T \delta q \), where \( \delta x(t) \) means the variation of the function \( x(t) \).

### III. Control System Design

As shown in Fig. 2, only the first quadrotor knows the transportation destination determined by a virtual leader. Hence, a finite-time observer is designed for the quadrotors to estimate the virtual leader’s information synchronously. Then, the disturbances acting on the quadrotors from the deformation of the flexible beam are analyzed considering the possible transverse and torsional deformations. Finally, a hierarchical distributed controller with the compensation of the disturbances from the transverse and torsional deformations and some unmodeled terms is designed.

#### A. Finite-time Leader Observer

![Communication graph](image)

The desired transportation position and yaw angle determined by the virtual leader are represented by \([X_d, Y_d, Z_d, \psi_d]^T\) and \(\psi_d\), respectively. Denote \(\alpha_d = [X_d, Y_d, Z_d, \psi_d]^T\). Suppose that there exist three positive constants \(\gamma_1, \gamma_2 \) and \(\gamma_3\) such that \(\|\alpha_d\| < \gamma_1\), \(\|\tilde{\alpha}_d\| < \gamma_2\) and \(\|\tilde{\tilde{\alpha}}_d\| < \gamma_3\) hold.

Hence, the following finite-time distributed observer is designed:

\[
\begin{align*}
\dot{\alpha}_i &= -r_1 \text{sign} \xi_1 \sum_{j=1}^{N} \alpha_{ij} (\alpha_i - \alpha_j) + b_i (\alpha_i - \alpha_d) \\
\dot{\beta}_i &= -r_2 \text{sign} \xi_2 \sum_{j=1}^{N} \beta_{ij} (\beta_i - \beta_j) + b_i (\beta_i - \beta_d) \\
\dot{\gamma}_i &= -r_3 \text{sign} \xi_3 \sum_{j=1}^{N} \gamma_{ij} (\gamma_i - \gamma_j) + b_i (\gamma_i - \gamma_d)
\end{align*}
\]

where \(i = 1, 2, 3, \ldots, n\), \(\alpha_i = [\alpha_{i1}, \alpha_{i2}, \alpha_{i3}, \alpha_{i4}]^T \in \mathbb{R}^4\), \(\beta_i = [\beta_{i1}, \beta_{i2}, \beta_{i3}, \beta_{i4}]^T \in \mathbb{R}^4\), \(\gamma_i = [\gamma_{i1}, \gamma_{i2}, \gamma_{i3}, \gamma_{i4}]^T \in \mathbb{R}^4\), \(r_1 > 0, r_2 > 0, r_3 > 0\), \(r_4 > 0, r_5 > 0, r_6 > 0\), \(0 < \xi_1 < 1, 0 < \xi_2 < 1, 0 < \xi_3 < 1\).

The definitions of \(a_{ij}\) and \(b_i\) can be found in Section II-A.

For the quadrotors with no access to the leader’s information, \(b_i = 0\) holds, i.e., \(b_i(\alpha_i - \alpha_d), b_i(\beta_i - \beta_d)\) and \(b_i(\gamma_i - \gamma_d)\) will be removed from the observer in Eq. (2). The initial values of \(\alpha_i, \beta_i, \gamma_i\) are chosen as \([X_0(i), Y_0(i), Z_0(i), \psi_0(i)]^T\), \([X_0(0), Y_0(0), Z_0(0), \psi_0(0)]^T\) and \(0_{1,1}\), respectively.

**Lemma 1:** The distributed observer in Eq. (2) is globally finite-time convergent under the communication graph in Fig. 2 with \(r_1 > 0, r_2 > \gamma_1, r_3 > 0, r_4 > \gamma_2, r_5 > 0, r_6 > \gamma_3, 0 < \xi_1 < 1, 0 < \xi_2 < 1, 0 < \xi_3 < 1\).

Based on the estimation of the information of the virtual leader using the observer in Eq. (2), for the \(i\)th quadrotor, the desired position, velocity and acceleration are \(\hat{r}_{di} = [\hat{\alpha}_{i1}, (i - 1)l \cos \alpha_{i4}, \alpha_{i2} + (i - 1)l \sin \alpha_{i4}, \alpha_{i3}]^T\), \(\hat{r}_{di} = [\hat{\beta}_{i1}, (i - 1)l \beta_{i4}, \hat{\beta}_{i2} + (i - 1)l \beta_{i4}, \hat{\beta}_{i3}]^T\) and \(\hat{r}_{di}^T = [\hat{\gamma}_{i1}, (i - 1)l \gamma_{i4}, \hat{\gamma}_{i2} + (i - 1)l \gamma_{i4}, \hat{\gamma}_{i3}]^T\), \(\hat{r}_{di}^T = [\hat{\psi}_{i1}, (i - 1)l \psi_{i4}, \hat{\psi}_{i2} + (i - 1)l \psi_{i4}, \hat{\psi}_{i3}]^T\), respectively.

The reference angle, angular velocity and angular acceleration in yaw direction are \(\hat{\psi}_{di} = \alpha_{i4}, \hat{\psi}_{dij} = \beta_{i4}\) and \(\hat{\psi}_{dij}^T = \gamma_{i4}\). Obviously, after the convergence of the distributed observer in Eq. (2), \(\hat{r}_{di}, \hat{r}_{dij}, \hat{r}_{dij}^T, \hat{r}_{di}^T, \hat{r}_{di}^T\) and \(\hat{r}_{dij}^T\) will equal to \(r_{di} = [X_d + (i - 1)l \cos \psi_d, Y_d + (i - 1)l \sin \psi_d, Z_d]^T, \hat{r}_{di}, \hat{r}_{dij}, \hat{r}_{dij}^T, \hat{r}_{di}^T, \hat{r}_{di}^T\), respectively.

#### B. Transverse and Torsional Deformations

![Transverse deformation](image)

1) **Transverse Deformation of a Beam with Two Quadrotors:** Each part of the flexible beam in Fig. 1 is held by two quadrotors, i.e., the states of these two quadrotors will affect the deformation of this part of the flexible beam. Hence, this subsection will focus on the system in Fig. 3. The free transverse vibration of such a system under the Earth’s gravity field can be described by \(\rho \frac{d^2 w_i}{dy^2} + EI \frac{d^4 w_i}{dy^4} = -\rho g\), where \(w_i\) represents the transverse deformation of the beam. Suppose that the movement of the quadrotor is much slower than the beam vibration, i.e., the beam will converge to the stable configuration quickly with various boundary conditions. Hence, one has \(\frac{d^2 w_i}{dy^2} = -\frac{\rho g}{2AEI}\), whose solution is

\[
w_i = c_1 + c_2 y + c_3 y^2 + c_4 y^3 = -\frac{\rho g}{2AEI} y^4
\]

where \(c_1, c_2, c_3 \) and \(c_4\) are constants determined by the boundary conditions. For the system in Fig. 3, the boundary conditions are \(w_i(0) = Z, w_i(l) = Z_{i+1}, \frac{dw_i}{dy}(0) = \phi_i \) and \(\frac{dw_i}{dy}(l) = \phi_{i+1}\). Substituting Eq. (3) into the boundary conditions yields \(A_1[c_1, c_2, c_3, c_4]^T = [Z_i, Z_{i+1} + \frac{\rho g l^3}{2AEI}, \phi_i, \phi_{i+1} + \frac{\rho g l^3}{6EI}]^T\), where
\[ A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \]

Therefore, \([c_1, c_2, c_3, c_4]^T = A_1^{-1} [Z_i, Z_{i+1}, \ldots] \) is defined in Eq. (5). Consider the generalized energy of the translational movement of the \(i\)th quadrotor \(E_1 = V_1\), where \(V_1\) is the kinetic energy of the quadrotor.

The RBF NN is used to estimate \(d_{p,i} = d_{p,i} - [0, 0, f_{i,i} + f_{i-1,i}]^T\), where the definition of \(f_{i,i}\) can be found in the last paragraph in Section III-B1. Note that both \(f_{0,1}\) and \(f_{n,n}\) equal to zero. Suppose that the estimated output of the NN is \(\hat{d}_{p,i} = W_d^T h(\bar{x})\), where \(W_d \in \mathbb{R}^{n \times 1}\) is the estimate of the ideal weight matrix \(W_d\) satisfying \(\hat{d}_{p,i} = W_d^T h(\bar{x}) + \epsilon_1\), where \(\epsilon_1\) is bounded by \(c_1^T\), i.e., \(||\epsilon_1|| \leq c_1^T\). Hence, \(\bar{x}\) is chosen as \(\bar{x} = [\bar{r}^T_{qi}, \bar{r}^T_{q(i+1)}, \bar{\psi}^T_i, \bar{\psi}^T_{i+1}]^T\). \(W_d\) is updated by

\[ W_d = \Gamma_1^T h(\bar{x}) \bar{s}_{p,i}^T \]

(6)

where \(\Gamma_1\) is a positive definite matrix.

Design the virtual position controller as

\[ \hat{f}_i = -\eta_p s_{p,i} + m_q g z_i - [0, 0, f_{i,i} + f_{i-1,i}]^T + m_q \hat{r}_{d_{qi}} \]

(7)

where \(\eta_p\) is a positive definite matrix. \(f_{i,i} + f_{i-1,i}\) is used to compensate for the disturbance force acting on quadrotor \(i\) due to the transverse deformation. Therefore, the closed-loop system can be written as

\[ m_q \ddot{s}_{p,i} = -\eta_p s_{p,i} + \hat{d}_{p,i} - \hat{d}_{p,i} + \Delta_{p,i} \]

(8)

where \(\Delta_{p,i}\) is \(m_q \ddot{s}_{p,i} - m_q \eta_q \hat{r}_{di} + m_q \hat{r}_{d_{qi}} + m_q \eta_q \hat{r}_{d_{qi}}\). We will equal to zero after the convergence of the observer in Eq. (2).

Theorem 1: Consider the translational dynamics of the quadrotor in Eq. (4). If the controller in Eq. (7) with the distributed observer in Eq. (2) and the updating law in Eq. (6) is applied, the position tracking error is ultimately uniformly bounded.

Proof: In the first step, the stability of the closed-loop system in the case that the observer has converged, i.e., the centralized case is discussed. It will be shown that the system will not go to infinity in a finite time in the second step. Step A: Suppose that the distributed observer in Eq. (2) has converged, i.e., each quadrotor knows the leader’s information. The closed-loop system in Eq. (8) becomes

\[ m_q \ddot{s}_{p,i} = -\eta_p s_{p,i} + W_d^T h(x) - \hat{W}_d^T h(\bar{x}) + \epsilon_1 \]

(9)

For the above equation, choose the Lyapunov function as

\[ V_1 = \frac{1}{2} m_q^T \dot{s}_{p,i}^T + \frac{1}{2} \text{trace}((\bar{W}_d - W_d)^T \Gamma_{-1}^T (\bar{W}_d - W_d)) \]

(10)

The time derivative of \(V_1\) along Eq. (9) is

\[ \dot{V}_1 = -\eta_q^T \dot{s}_{p,i} + s_{p,i}^T \epsilon_1 \leq -\|s_{p,i}\| \langle 2\eta_q (\eta_q)^T s_{p,i} \rangle - \epsilon_1 \]

Therefore, the system will converge to the set \(\{s_{p,i}, \|s_{p,i}\| \leq \epsilon_1 / \eta_q (\eta_q)^T s_{p,i} \}\). Furthermore, essentially, the system governed by \(\bar{r}_d - r_{di} + \eta_q (r_{qi} - r_{di}) + \eta_2 \int_0^t (r_{qi} - \bar{r}_{di}) dt\) is a damped oscillation system, i.e., it is asymptotically stable. According to the definition of \(s_{p,i}\), \(r_{qi} - r_{di} + \eta_2 \int_0^t (r_{qi} - r_{di}) dt\) in the centralized case, the position tracking error \(r_{qi} - r_{di}\) is ultimately uniformly bounded with a bounded \(s_{p,i}\).

Step B: Before the convergence of the distributed observer (2), \(s_{p,i}\) is defined in Eq. (5). Consider the generalized energy of the translational movement of the \(i\)th quadrotor \(E_1 = V_1\),
where $V_1$ is defined in Eq. (10). It is clear that $E_1 \geq \frac{1}{2} m_q \|s_{p,i}\|^2_2$. Along Eq. (8), $\dot{E}_1$ reads $\dot{E}_1 = -s_{p,i}^T \eta_{ps} s_{p,i} + s_{p,i}^T \Gamma_{p,i}$. Due to the finite-time stability of the observer (2) shown in Lemma 1, the term $\Delta_{p,i}$ is bounded. Suppose there exists a constant $c_i^*$ such that $\|\Delta_{p,i}\|_2 \leq c_i^*$. Hence, $\dot{E}_1$ satisfies $\dot{E}_1 \leq -\lambda_{\min}(\eta_{ps})\|s_{p,i}\|^2_2 + \lambda_{\min}(\eta_{ps})\|s_{p,i}\|^2_2 + \frac{1}{2}\|s_{p,i}\|^2_2 + (c_i + c_i^*)^2 c_i^2 E_i + c_i^3$, where $c_i^2 = \frac{1}{2}(c_i + c_i^*)^2$. Integrating both sides of $\dot{E}_1 \leq c_i^2 E_i + c_i^3$ yields $E_1 \leq -\frac{c_i^2}{c_i^3} + C\|E(0)\| + c_i^3$, which implies that the general energy $E_1$ is bounded in a finite time.

Therefore, the closed-loop system governed by Eq. (8) cannot escape in a finite time. After the convergence of the finite-time observer in Eq. (2), the distributed control problem becomes a centralized one and as shown in Step A, the centralized controller can ensure the boundedness of the position tracking errors. Based on the separation principle between the observer and controller [14, 16], one can conclude that the conclusion in Step A still holds for the distributed controller in Eq. (7) as time goes to infinity.

2) Inner Control Loop: To realize the position control command $f_i = [f_{i,1}, f_{i,2}, f_{i,3}]^T$ in Eq. (7), for the $i$th quadrotor, let the thrust force be $f_i = \|f_i\|_2$ and define the desired roll and pitch angles as $\phi_{di} = \arcsin \left( \frac{f_{i,1} \cos \psi_{a,i} + f_{i,2} \sin \psi_{a,i}}{\|f_i\|_2} \right)$ and $\theta_{di} = \arctan \left( \frac{f_{i,1} \cos \psi_{a,i} - f_{i,2} \sin \psi_{a,i}}{f_{i,3}} \right)$, respectively. Denote $\hat{\Psi}_{di} = [\phi_{di}, \theta_{di}, \psi_{di}]^T$, $\hat{\Psi}_{di} = [\phi_{di}, \theta_{di}, \psi_{di}]^T$ and $\tilde{\Psi}_{di} = [\phi_{di}, \theta_{di}, \psi_{di}]^T$.

Define the following sliding variable

$$s_{a,i} = \hat{\Psi}_i - \hat{\Psi}_{di} + \eta_3 (\hat{\Psi}_i - \hat{\Psi}_{di}) + \eta_1 \int_0^t (\Psi_i - \tilde{\Psi}_{di}) dt \quad (11)$$

where $\Psi_i$ is defined in the paragraph after Eq. (1). $\hat{\Psi}_{di}$ and $\hat{\Psi}_{di}$ are given in the last paragraph in Section III-A, and $\eta_3$ and $\eta_1$ are two positive definite diagonal matrices.

Similar to the position control, suppose that $d_{a,i} = d_{a,i} - J_q (\hat{\theta}_{di}, \hat{\psi}_{di}, 0)^T - [\tau_{1,i,i} + \tau_{1,i,i-1}, \tau_{2,i,i}, \tau_{2,i,i-1}]0^T$ can be written as $d_{a,i} = W_{ad} h_{a}(\hat{x}) + \epsilon_{2}$, where $W_{ad}$ is the ideal weight matrix, $h_{a}(\hat{x})$ is the vector of Gaussian activation functions and $\epsilon_{2}$ is bounded by $c_2^*$, i.e., $\|\epsilon_{2}\|_2 \leq c_2^*$. Note that the definitions of $\tau_{1,i,i}, \tau_{1,i,i-1}, \tau_{2,i,i}$ and $\tau_{2,i,i-1}$ can be found in Section III-B and $\tau_{1,n,n} = \tau_{1,0,n} = \tau_{2,n,n} = \tau_{2,1,0} = 0$ holds. The estimate of $W_{ad}$ is represented by $\hat{W}_{ad}$, which is updated by the following law

$$\hat{W}_{ad} = \Gamma_{2} h_{a}(\hat{x}) s_{a,i} \quad (12)$$

where $\Gamma_2$ is a positive definite matrix.

Design the following attitude controller

$$\tau_i = J_q(0, 0, \psi_{di}^\circ)^T + \hat{\Psi}_i \times (J_q \hat{\Psi}_i - \eta_3 J_q \hat{\Psi}_i - \hat{\Psi}_{di})$$

$$-\eta_1 J_q (\hat{\Psi}_i - \hat{\Psi}_{di}) - W_{ad} h_{a}(\hat{x}) - \eta_6 s_{a,i}$$

$$- [\tau_{1,i,i} + \tau_{1,i,i-1}, \tau_{2,i,i} + \tau_{2,i,i-1}, 0]^T \quad (13)$$

where $\eta_6$ is a positive definite matrix and $[\tau_{1,i,i} + \tau_{1,i,i-1}, \tau_{2,i,i} + \tau_{2,i,i-1}, 0]^T$ is used to compensate for the disturbance torques because of the transverse and torsional deformations.

Therefore, the closed-loop system is governed by

$$J_q s_{a,i} = -\eta_6 s_{a,i} + \hat{d}_{a,i} - W_{ad} h_{a}(\hat{x}) + \Delta_{a,i} \quad (14)$$

where $\Delta_{a,i} = J_q \hat{\Psi}_{di} - J_q \hat{\Psi}_{di} - \eta_3 J_q \hat{\Psi}_{di} - \eta_3 J_q \hat{\Psi}_{di}$. It should be noted that $\Delta_{a,i}$ can be rewritten as $\Delta_{a,i} = J_q(0, 0, \psi_{di}^\circ)^T + \eta_3 J_q[0, 0, \psi_{di}^\circ - \psi_{di}]^T + \eta_3 J_q[0, 0, \psi_{di}^\circ - \psi_{di}]^T$. According to Lemma 1, $\Delta_{a,i}$ will go to zero after the convergence of the distributed observer (2).

Theorem 2: For the attitude dynamics of the $i$th quadrotor governed by Eq. (1) under the controller in Eq. (13) with the distributed observer in Eq. (2) and the updating law in Eq. (12), the attitude tracking error is ultimately uniformly bounded.

The proof of Theorem 2 is similar to that of Theorem 1.

IV. NUMERICAL SIMULATIONS

In this section, numerical simulations are presented for the transportation of a flexible beam using two quadrotors to verify the effectiveness of the proposed position controller (7) and attitude controller (13), i.e., $n = 2$ in the simulation. The dynamic model built in Section II-C with $n_1 = 2$ and $n_2 = 1$ is used as the control plant, i.e., the first two bending modes and the fundamental torsional mode of a cantilever beam are adopted to discretize the flexible beam. The system parameters $m_q$, $J_q$, $\rho$, $I$, $E$ and $G J P$ are $1.121$ kg, $\operatorname{diag}(0.01, 0.0082, 0.0148)$ kgm$^2$, $0.079$ kg/m, $0.80$ m, $0.152$ Nm$^2$ and $55.8$ Nm. In the RBF NNs in the simulation, the neuron number in the hidden layer is 20. The center vector $\mu_i$ is chosen as a vector with uniformly distributed random numbers from $-0.5$ to $0.5$. The same width for each Gaussian function is selected as $0.5$. The control parameters $r_i$, $\xi_i$, $\eta_l$, $\eta_2$, $\eta_3$, $\eta_4$, $\eta_5$ and $\eta_6$ are chosen as $1$, $0.5$, $\operatorname{diag}(3, 3, 5)$, $\operatorname{diag}(3, 3, 25)$, $\operatorname{diag}(2, 2, 3)$, $\operatorname{diag}(0.01, 0.01, 0.01)$, $\operatorname{diag}(2.5, 2.5, 0.09)$ and $\operatorname{diag}(0.11, 0.11, 0.055)$.

At the initial time, the generalized coordinate vector of the system model in Section II-C equals to $\theta_{0x1}$, i.e., two quadrotors stay at $[0, 0, 0]^T$ and $[0, 0, 0]^T$, respectively, with zero roll, pitch and yaw angles and zero beam deformation. The trajectory and yaw angle of the virtual leader are chosen as $[X_d, Y_d, Z_d]^T = [X_d(t), Y_d(t), Z_d(t)]^T$ and $\psi_d = 0$, where $X_d(t) = \begin{cases} 0, & 0 \leq t \leq 20, \\ 0.24 \times (t - 20), & 20 < t < 25, \\ 1.2, & t \geq 25 \end{cases}$, $Y_d(t) = \begin{cases} \begin{array}{ll} 0.3 \times (t - 30), & 30 \leq t < 35, \\ 1.2, & t \geq 35 \end{array} \end{cases}$ and $Z_d(t) = \begin{cases} 1, & t < 5 \\ 1 + 0.2 \times (t - 5), & 5 \leq t < 9 \\ 1.8, & 9 \leq t - t_i < 11 \\ 1.8 - 0.1 \times (t - 11), & 11 \leq t < 15 \\ 1.4, & t \geq 15 \end{cases}$.

The blue dashed curves in Fig. 5 represent the virtual leader’s trajectories in X, Y and Z directions. Note that the virtual leader’s information is fed into the finite-time observer in Eq.
With a distributed observer of the virtual leader’s information and the compensation of these disturbances, a hierarchical formation flying controller was designed to complete the aerial transportation task. Numerical simulation results based on the nonlinear dynamics of the transportation system indicated that the proposed controller can achieve better performance than a PID controller.

**REFERENCES**


