2020 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)
October 25-29, 2020, Las Vegas, NV, USA (Virtual)

Minimally Disruptive Connectivity Enhancement for Resilient Multi-Robot Teams

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***Abstract*— In this work, we focus on developing algorithms to maintain and enhance the connectivity of a multi-robot system with minimal disruption to the primary tasks that the robots are performing. Such algorithms are useful for collaborating robots to be resilient to reduction in connectivity of the communication graph of the robot team when robots can arrive or leave. These algorithms are also useful in a supervisory control setting when an operator wants to enhance the connectivity of the robot team. In contrast to many existing works that can only maintain the current connectivity of the multi-robot graph, we propose a generalized connectivity control framework that allows for reconfiguration of the multi-robot system to provably satisfy any connectivity demand, while minimally disrupting the execution of their original tasks. In particular, we propose a novel $k-$Connected Minimum Resilient Graph (k-CMRG) algorithm to compute an optimal $k-$connectivity graph that minimally constrains the robots' original task-related motion, and employ the Finite-Time Convergence Control Barrier Function (FCBF) to enforce the pairwise robot motion constraints defined by the edges of the graph. The original controllers are minimally modified to drive the robots and form the k-CMRG. We demonstrate the effectiveness of our approach via simulations in the presence of multiple tasks and robot failures.**

I. Introduction

Multi-robot systems are widely studied for their capability of accomplishing complex tasks through cooperative behaviors, e.g. environmental sampling [1], area coverage [2], search and rescue [3]. In many large-scale multi-robot applications, the robotic team executes simultaneously multiple behaviors or sequences of behaviors [4]–[6] with various task-prescribed controllers in real time to increase efficiency in parallel tasks. The success of inter-robot coordination often requires continuous connectivity between robots [7]–[11]. Due to limited communication range of the robots, connectivity is maintained by constraining the inter-robot distance during their task execution, so that the robots stay connected as one component to enable local information sharing and interaction. Moreover, given the growing scale of the multi-robot team, the increasing number of expected robot failures necessitates the recent research on robust connectivity maintenance [12]–[15], e.g. maintaining an initially connected $k-$node connectivity graph [15], so that the failure of less than k robots will not disconnect the multi-robot network, or developing heuristic-based secondary connectivity controller [13], [14] to increase the vulnerable neighbors' node degrees to improve the network robustness.

New challenges arise for multi-robot applications with parallel tasks [5] in adversarial scenarios. For example, the execution of multiple and potentially conflicting tasks simultaneously could easily cause multi-robot network disconnection. In this case, the connectivity maintenance behavior could dominate over the original robot controllers and largely impacts the original task execution. On the other hand, in the presence of continuous robot failures due to adversarial situations, the network robustness will keep decreasing and could eventually lead to network disconnection. Thus, it demands a resilient connectivity controller that respects the original multi-robot task-related controllers with guaranteed fault-tolerant and resilient connectivity maintenance, so that the robotic team could always maintain, recover, and increase the network connectivity in presence of continuous robot removals while progressing over their original tasks. This is particularly challenging for most existing work since (a) the resilience and robustness of the multi-robot network leads to increasing complexity over conventional connectivity control methods [8], [9], [16], [17] due to the possible discontinuity from dynamic topology changes as pointed out in [18], (b) conventional connectivity metrics such as algebraic connectivity is not suitable to explicitly model the network robustness as found by [13], [14], (c) there is often no optimality guarantee over the imposed connectivity constraints for original robot tasks [4], [19], [20] nor the perturbation from the connectivity controller to the robot original controllers [13], [14], [21], and (d) the network robustness is maintained in absence of robot failures [13], [15] and hence could be vulnerable to increasing number of robot failures over time.

Motivated by the challenges, in this paper we aim to develop provably optimal algorithms for minimally disruptive and resilient connectivity maintenance for a team of connected robots. We assume the robots have been provided with their original task-related controllers and seek to revise their controllers as necessary to achieve the desired resilient network connectivity and avoid collisions between robots and with obstacles. In particular, we propose a minimally disruptive resilient connectivity maintenance framework that, by inputting *any* desired value k of graph connectivity, the framework will first compute the provably optimal $k-$node connected minimum resilient graph (k-CMRG) whose edges invoke *min-size* pairwise connectivity constraints *least violated* by the robot original controllers. With the rendered

This work was funded by the AFOSR awards FA9550-18-1-0097, FA9550-15-1-0442 and Department of Agriculture (DOA) 2017-6700-726152.

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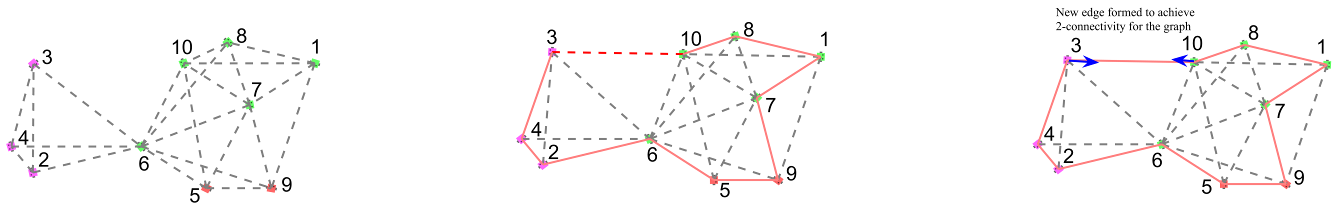


(a) Initial 1-connected (k=1) graph

(b) Improved 2-connected (k=2) graph with red existing connectivity edges to preserve and one augmented edge (red dash edge) to form

(c) Robots perform their original task while ensuring the existing red edges stay connected and the augmented edge(s) are formed (i.e. robots 3 and 10 get closer)

Fig. 1: Simple example of resilient connectivity maintenance problem, where 10 robots are executing their original behaviors while preserving and forming the red connectivity edges defined by k-CMRG to achieve the desired connectivity (e.g. k=2 here). Connectivity edges (gray dash line) exist when two robots are within the limited communication range. Red solid edges are selected from existing gray connectivity edges and augmented with the selected red dash edge from non-exist connectivity edges to compose a k-CMRG with desired graph connectivity.

optimal pairwise connectivity constraints, we employ the Finite-time Convergence Barrier Function (FCBF) from [4] to map the invoked pairwise spatial connectivity constraints to those over the robot original controllers, and minimally modify those controllers in the context of quadratic programming to respect the original tasks. Note that in [4] the connectivity controllers are optimized with predefined connectivity constraints by the user and does not consider resilience, while here we are optimizing both the *resilient* connectivity constraints to activate as well as the revision to the original controller. Also in contrast of our previous work [15] that requires the robots are $k-$node connected as initial condition, here we relax that assumption and allow the robots to achieve arbitrary required resilience connectivity over time, e.g. the robot team reconfigures to build new connections between robots and achieve higher connectivity to recover the decreased connectivity due to the loss of robots, or to increase connectivity and thus improve robustness.

Our paper presents the following contributions: (1) a generalized resilient connectivity maintenance framework that jointly optimizes both the topological resilient connectivity graph and the constrained robot motions to minimally disrupt the original robot tasks, (2) a novel $k-$CMRG method to compute the optimal weighted $k-$node connected resilient graph for arbitrary initially connected multi-robot graph, imposing *least* connectivity constraints to the robots, (3) theoretical analysis and proof of the optimality of our k-CMRG with guaranteed, user-specified network resilient connectivity in presence of continuous robot failures.

II. Problem Formulation

Consider a robotic team $\mathcal{S}$ consisting of n mobile robots in a planar space, with the position and single integrator dynamics of each robot $i \in \{1, \dots, n\}$ denoted by $x_i \in \mathbb{R}^2$ and $\dot{x}_i = u_i \in \mathbb{R}^2$ respectively. Each robot can connect and communicate directly with other robots within its spatial proximity. The communication graph of the robotic team is defined as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each node $v \in \mathcal{V}$ represents a robot. If the spatial distance between robot $v_i \in \mathcal{V}$ and robot $v_j \in \mathcal{V}$ is less or equal to the communication radius $R_c \in \mathbb{R}$ (i.e. $\|x_i - x_j\| \leq R_c$), then we assume the two can communicate and edge $(v_i, v_j) \in \mathcal{E}$ is undirected (i.e. $(v_i, v_j) \in \mathcal{E} \Leftrightarrow (v_j, v_i) \in \mathcal{E}$).

We assume the robotic team has been tasked with m simultaneous behaviors, partitioning the set of robots into m sub-groups. To simplify our discussion, we assume the sub-group partitions and behavior controllers are given or already derived from other multi-robot task allocation algorithms, namely, each robot i has been assigned to a sub-group with some behavior-prescribed controller $u_i = \hat{u}_i$. We also assume the current communication/connectivity graph $\mathcal{G}$ for the robots is connected as one component. Here the multi-robot network resilience is quantified by the network connectivity defined as follows [22].

Definition 1. *(k-node connected graph) A connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be k-node connected (or k-connected) if it has more than k nodes and remains connected whenever fewer than k nodes are removed.*

The objective of our paper is, given any user defined desired connectivity $k \in \mathbb{R}^+$, how to develop control laws for the robots to achieve and maintain it over time, even if the current connectivity graph is not $k-$node connected. This differentiates our work from most of the connectivity maintenance literature [4], [5], [8], [15], [19], [20] and thus provides greater freedom to enable flexible resilient connectivity maintenance. In the rest of the paper, we will use $k-$connectivity to represent $k-$node connectivity. Then we would like to enforce such constraint as robots execute their behavior-prescribed controllers, so that the resulting time-varying connectivity graph $\mathcal{G}$ becomes and stays $k-$connected at all time. It is straightforward that $n \geq k+1$ should be followed in order for the problem to be solvable [23], [24]. In presence of the above connectivity constraints as well as the physical constraints of the robots such as inter-robot collision avoidance and velocity limits, each robot i may have to modify their primary task-related controller $\hat{u}_i$ to accommodate the constraints. To that end, the objective is to 1) invoke active constraints to follow (particularly the connectivity constraints imposed between pair-wise robots), such that the modification to the primary controller is minimum for the robotic team, and 2) compute the modified controllers for robots task execution. In the remainder of this section, we will discuss the formulation of the mentioned constraints in the form of Control Barrier Function (CBF) [19], [25], [26] and Finite-Time Convergence Control Barrier Function (FCBF) [4] on the controllers followed by the optimization problem formulation.

A. Safety Constraints using Safety Barrier Certificates

During movements of multi-robot systems, the robots should avoid collisions with each other to remain safe. Consider the joint robot states $\mathbf{x} = \{x_1, \dots, x_n\} \in \mathbb{R}^{2n}$ and define the minimum safe distance as R_s for any pair-wise inter-robot collision avoidance constraint. We have the following condition defining the safe set of $\mathbf{x}$.

$$\begin{aligned} h^s_{i,j}(\mathbf{x}) &= \|x_i - x_j\|^2 - R_s^2, \qquad \forall i > j \\ \mathcal{H}^s_{i,j} &= \{\mathbf{x} \in \mathbb{R}^{2n} : h^s_{i,j}(\mathbf{x}) \geq 0\} \end{aligned} \tag{1}$$

The set of $\mathcal{H}^s_{i,j}$ indicates the safety set from which robot i and j will never collide. For the entire robotic team, the safety set can be composed as follows.

$$\mathcal{H}^s = \bigcap_{\{v_i, v_j \in \mathcal{V} : i > j\}} \mathcal{H}^s_{i,j} \tag{2}$$

[27] proposed the safety barrier certificates $\mathcal{B}^s(\mathbf{x})$ using control barrier functions (CBF) [25] that map the constrained safety set (2) of $\mathbf{x}$ to the admissible joint control space $\mathbf{u} \in \mathbb{R}^{2n}$. The result is summarized as follows.

$$\mathcal{B}^s(\mathbf{x}) = \{\mathbf{u} \in \mathbb{R}^{2n} : \dot{h}^s_{i,j}(\mathbf{x}) + \gamma h^s_{i,j}(\mathbf{x}) \geq 0, \forall i > j\} \tag{3}$$

where γ is a user-defined parameter to confine the available sets. It is proven in [27] that the forward invariance of the safety set $\mathcal{H}^s$ is ensured as long as the joint control input $\mathbf{u}$ stays in set $\mathcal{B}^s(\mathbf{x})$. In other words, the robots will always stay safe if they are initially inter-robot collision free and the control input lies in the set $\mathcal{B}^s(\mathbf{x})$. Note that at any time point t with known current robot states $\mathbf{x}(t)$, the constrained control space in (3) corresponds to a class of linear constraints over pair-wise control inputs u_i and u_j for $\forall i > j$. Note that static obstacles may also be modelled in the same manner if treated as robots with zero velocity.

B. Connectivity Constraints using Finite-Time Control Barrier Function

Similar to safety barrier certificates for collision avoidance, pairwise connectivity constraints can also be mapped to the admissible set for control input in the same manner [19]. However, the forward invariance from CBF requires the system already in the desired set, e.g. robots are initially collision free and so to stay safe. To enforce connectivity constraints used to form new edges, [4] proposed the Finite-Time Convergence Control Barrier Functions (FCBF) that could drive the robots from outside to the admissible set and stay inside the desired states. This has been applied to form and then preserve new connectivity edges predefined by the tasks [4]. Here we briefly introduce the mapping from a particular pairwise connectivity constraint to the admissible set for controllers using FCBF.

To enforce a connectivity constraint between pair-wise robots i and j to limit the inter-robot distance not larger than communication range R_c, we have the following condition.

$$\begin{aligned} h^c_{i,j}(\mathbf{x}) &= R_c^2 - \|x_i - x_j\|^2 \\ \mathcal{H}^c_{i,j} &= \{\mathbf{x} \in \mathbb{R}^{2n} : h^c_{i,j}(\mathbf{x}) \geq 0\} \end{aligned} \tag{4}$$

The set of $\mathcal{H}^c_{i,j}$ indicates the feasible set on $\mathbf{x}$ from which robot i and j will never lose connectivity. Then for any connectivity graph $\mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c)$ to enforce, the corresponding constrained set can be composed as follows.

$$\mathcal{H}^c(\mathcal{G}^c) = \bigcap_{\{v_i, v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}^c\}} \mathcal{H}^c_{i,j} \tag{5}$$

The connectivity barrier certificates are hence defined as follows using FCBF [4] that indicates another class of linear constraints over pair-wise control inputs u_i and u_j for $(v_i, v_j) \in \mathcal{E}^c$ at any time point t.

$$\begin{aligned} \mathcal{B}^c(\mathbf{x}, \mathcal{G}^c) = \{\mathbf{u} \in \mathbb{R}^{2n} : &\dot{h}^c_{i,j}(\mathbf{x}) + \gamma \cdot \text{sign}(h^c_{i,j}(\mathbf{x})) \cdot |h^c_{i,j}(\mathbf{x})|^\rho \geq 0, \\ &\forall (v_i, v_j) \in \mathcal{E}^c\} \end{aligned} \tag{6}$$

where $\rho \in [0, 1)$ determines how fast the system is driven towards the set of $\mathcal{H}^c_{i,j}$. It has been proved in [4] that for any initial condition $\mathbf{x}_0$, any controller subject to (6) will drive the system to the set $\mathcal{H}^c$ in a finite time bounded by $T = \frac{|h^c_{i,j}(\mathbf{x})|^{1-\rho}}{\gamma(1-\rho)}$. This property ensures that for any pairwise connectivity constraint that is not currently satisfied, we can allocate a time period larger than T for the constraint to render the new connectivity edges. Note that the FCBF takes as inputs a given graph $\mathcal{G}^c$ which is predefined in [4]. We will use this sub-routine to enforce the construction of desired $k-$CMRG in our resilient connectivity maintenance framework in the following.

C. Objective Function

Consider that a task-related primary behavior control input $\hat{u}_i \in \mathbb{R}^2$ has been computed for each robot i before considering the mentioned constraints. The objective is to minimally modify the primary controllers subject to connectivity and safety constraints. Different from other optimization-based framework with CBF [19] or FCBF [4] with predefined connectivity constraints, here we extend to the resilient connectivity maintenance framework, where the robots are optimizing both the $k-$connectivity constraints to enforce and the controllers to revise. With the defined forms of constraints in (3) and (6), we formally define the *minimally disruptive resilient $k-$connectivity maintenance* problem with any given $k \leq n - 1$ at each time point t as follows.

$$\mathbf{u}^* = \underset{\mathcal{G}^c, \mathbf{u}}{\arg\min} \sum_{i=1}^{n} \|u_i - \hat{u}_i\|^2 \tag{7}$$

$$\text{s.t.} \quad \mathcal{G}^c = (\mathcal{V}, \mathcal{E}^c) \quad \text{is } k-\text{connected} \tag{8}$$

$$\mathbf{u} \in \mathcal{B}^s(\mathbf{x}) \bigcap \mathcal{B}^c(\mathbf{x}, \mathcal{G}^c), \quad \|u_i\| \leq \alpha_i, \forall i = 1, \dots, n \tag{9}$$

The above Quadratic Programming (QP) optimization problem is to find the optimal active $k-$connectivity graph $\mathcal{G}^c$ to enhance and the revised control inputs $\mathbf{u}^* \in \mathbb{R}^{2n}$ bounded by maximum velocity α_i for each robot, so that $k-$connectivity, safety and velocity constraints described in (8) and (9) are always guaranteed while ensuring minimally disruption to the primary controller as shown in (7). While the robust connectivity maintenance problem [15] has similar formulation, it requires $\mathcal{G}^c \subseteq \mathcal{G}$ and hence can only preserve the current connectivity and the subgraph from the existing graph. Here we relax this assumption and allow for connectivity enhancement with any desired connectivity k by forming new connectivity edges if necessary. This makes [15] a special case in our formulation when the current connectivity of $\mathcal{G}$ is already larger than desired connectivity k and there is no need to form new edges to increase connectivity. Note that as information regarding the primary task is not required other than $\hat{u}_i$, the objective of the original controller may not be guaranteed in form of (7) especially when it conflicts with connectivity or safety constraints, e.g. dispersing robots to different goal locations where robots get disconnected due to limited communication range. In this case, the objective of (7) first ensures constraints are satisfied at all time and then minimizes the deviation from original controller, e.g. dispersing robots towards assigned goal locations as much as possible while keeping them safe and k-connected.

The optimization problem in (7) can be decoupled into two dependent sub-problems: 1) compute provably optimal $k-$CMRG graph $\mathcal{G}^{c*} = \mathcal{G}^*_k$ that invokes *least violated* connectivity constraints over multi-robot behaviors, and then 2) solve the optimization problem (7) with the obtained optimal graph $\mathcal{G}^*_k$. In this way, it enables the robot team to form connectivity enhancement provably satisfying any demanded connectivity k while minimizing the disruption to their original tasks.

III. Maintaining Minimally Disruptive Resilient $k-$Connectivity

A. *Min-Size $k-$Node Connected Spanning Subgraph ($k-$NCSS)*

We consider the first sub-problem of computing optimal $k-$CMRG $\mathcal{G}^{c*} = \mathcal{G}^*_k(\mathcal{V}, \mathcal{E}^*_k)$ in (7) that introduces minimum $k-$connectivity constraints for any given connectivity demand k. Recall that each edge $(v_i, v_j) \in \mathcal{E}^c$ in a candidate graph $\mathcal{G}^c$ enforces one pair-wise linear constraint over primary control inputs $\hat{u}_i$ and $\hat{u}_j$ for robot i and j, as shown in (4). Thus it is straightforward that the optimal graph $\mathcal{G}^{c*}$ should have a minimum number of edges that satisfy k-connectivity.

Denote the connectivity of a graph by $\kappa(\cdot)$. Let us first consider the special case when $\kappa(\mathcal{G}) \geq k$, and then the $k-$CMRG boils down to finding a min-size $k-$Node Connected Spanning Subgraph ($k-$NCSS) with $\mathcal{G}^{c*} \subseteq \mathcal{G}$. This has been known as NP-hard for even $k = 2$ [24]. From graph theory, there exists a heuristic algorithmic framework, $k-$Node Connected Spanning Subgraph ($k-$NCSS) [23], [24] that finds the approximate min-size k-connected subgraph with uniform edge cost. Briefly, given an undirected connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and k where $k \leq \kappa(\mathcal{G})$, the min-size $k-$connected spanning subgraph $\mathcal{G}^*_k$ can be found by the following summarized algorithm.

Algorithm 1 Minimum-size $k-$node connected spanning subgraph ($k-$NCSS)

Input: $\mathcal{G}(\mathcal{V}, \mathcal{E})$, k
Output: $\mathcal{G}^*_k$
1: find a min-size $k-1$ edge cover $M \leftarrow \arg\min\{|M| : \deg_M(v) \geq k-1, \forall v \in \mathcal{V}, M \subseteq \mathcal{E}\}$
2: find an inclusionwise minimal edge set $F \subseteq \mathcal{E} \setminus M$ such that $(\mathcal{V}, M \cup F)$ is $k-$connected
3: **return** $\mathcal{G}^*_k \leftarrow (\mathcal{V}, M \cup F)$

With the Algorithm 1, we have the following Lemma regarding its known approximation of the derived $k-$connected spanning subgraph $\mathcal{G}^*_k$.

Lemma 2. *([23], [24]) Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a graph of node connectivity $\geq k$. Then the Algorithm 1 finds a $k-$node connected spanning subgraph $(\mathcal{V}, M \cup F)$ such that $|M \cup F| \leq (1 + \frac{1}{k})|\mathcal{E}_{opt}|$, where $|\mathcal{E}_{opt}|$ denotes the cardinality of the optimal solution.*

Hence Algorithm 1 provides a bounded solution to find a $k-$NCSS $\mathcal{G}^*_k \subseteq \mathcal{G}$ with minimum number of edges that could be used to define active pairwise $k-$connectivity constraints when $\kappa(\mathcal{G}) \geq k$. However, such solution could not handle the situation when $\kappa(\mathcal{G}) \leq k$ and it is more desirable to consider each edge differently due to their impact over the robots original controllers. For example, candidate connectivity constraint whose two robots are getting closer due to their original motion should be preferred, since maintaining such constraint will lead to less disruption over the original robot controllers. In the next section, we will propose a novel $k-$CMRG method to construct the optimal $k-$connected graph with any demanded connectivity $k \leq n-1$ and with consideration of the original robot controllers/motions.

B. *$k-$Connected Minimum Resilient Graph ($k-$CMRG)*

In general cases with arbitrary demanded connectivity k, each pairwise robots within the robotic team compose one candidate edge for determining $k-$CMRG $\mathcal{G}^{c*}$, which further increases the computation complexity of computing the optimal $k-$CMRG. Here we propose a new heuristic to evaluate any given candidate edge connecting pairwise robots $v_i, v_j \in \mathcal{V}$ as follows.

$$w_{i,j} = -\dot{h}^c_{i,j}(\mathbf{x}, \hat{u}_i, \hat{u}_j) - \gamma \cdot \mathrm{sign}(h^c_{i,j}(\mathbf{x})) \cdot |h^c_{i,j}(\mathbf{x})|^\rho \qquad (10)$$

This heuristic takes inspiration from the FCBF constraint in (6) and substitute with the original robot controllers $\hat{u}_i, \hat{u}_j$. Note that the smaller value of $w_{i,j}$ indicates forming/preserving the connectivity edges between v_i, v_j is less likely to be violated given the robot original controller. For example, $w_{i,j} < 0$ implies the FCBF constraint for preserving the corresponding edge is already satisfied by the original robot controllers without need of revision. Instead of checking for each pairwise candidate edges between any two robots in S, we augment the current connectivity graph with their weight defined by (10) and render a weighted connectivity graph $\hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with $w_{i,j} \in \mathcal{W}$. Next, we propose the following Algorithm 2 framework of our $k-$CMRG, a variant of Algorithm 1 with any connectivity demands k. For the rest of the paper, we use $k-$CMRG interchangeably to refer to the optimal $k-$connectivity graph or the algorithm to compute $k-$CMRG.

Algorithm 2 Outline of $k-$connected minimum resilient graph ($k-$CMRG)

Input: $\hat{\mathcal{G}}'(\mathcal{V}', \mathcal{E}', \mathcal{W}') \leftarrow \hat{\mathcal{G}}(\mathcal{V}, \mathcal{E}, \mathcal{W})$, k
Output: $\hat{\mathcal{G}}_k^*$
1: Expand $\mathcal{G}'$ by adding edges connecting robots with 2-hop neighbors until $\min\{\deg(v)\} \geq k$
2: find a min-size $k-1$ edge cover $M' = \arg\min_{M' \subseteq \mathcal{E}'} \quad \beta \cdot |M'| + \Sigma_{(v_i,v_j) \in M'}\{w_{i,j}\}$
3: find an inclusionwise minimal edge set $F' \subseteq \mathcal{E}' \setminus M'$ such that $(\mathcal{V}, M' \cup F')$ is $k-$connected, if not, expand $\hat{\mathcal{G}}'$ by adding edges connecting robots with 2-hop neighbors until F stay unchanged.
4: **return** $\hat{\mathcal{G}}_k^* \leftarrow (\mathcal{V}, M' \cup F')$

In Algorithm 2, there are several modifications compared to Algorithm 1. In Line 1 of Algorithm 2, it directly augmented 2-hop edges to the existing graph so that the minimum degree $deg(v)$ of each robot node is at least k. The reason lies in that for a $k-$connected graph, each robot has at least k edges and any one node will never be isolated with the removal of at most $k-1$ neighboring nodes.

In Line 2 of Algorithm 2, we redefine the min-size $(k-1)$ edge cover from Algorithm 1 to be M' by the following.

$$M' = \underset{M' \subseteq \mathcal{E}'}{\arg\min} \quad \beta \cdot |M'| + \Sigma_{(v_i,v_j) \in M'}\{w_{i,j}\} \tag{11}$$

where β is a pre-defined parameter and we assume $\beta >> 2 \cdot \Sigma_{\forall w_{i,j} \in \mathcal{W}'} |w_{i,j}|$, so that the selected edge cover set M' has minimum number of edges. And if there are multiple solutions with same number of edges, it will break ties by comparing the total weights and then select the one with minimum total weights. This implies least constrained edges to preserve with the original robot controllers. In the end (line 3), the inclusionwise minimal edge set is found by iterative expanding graph $\hat{\mathcal{G}}'$ until $\kappa(\hat{\mathcal{G}}') \geq k$. With the new condition above for finding $(k-1)$ edge cover set M', a new weighted $k-$connected minimum resilient graph ($k-$CMRG) can be derived as $\hat{\mathcal{G}}_k^* = (\mathcal{V}, \mathcal{E}_k', \mathcal{W}_k')$ with $\mathcal{E}_k' = M' \cup F' \subseteq \mathcal{E}'$. In particular, we have the following Theorem on bounded cardinality of edge set $\mathcal{E}_k'$ of the $k-$CMRG $\hat{\mathcal{G}}_k^*$.

Theorem 3. *Given weighted undirected graph $\hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ and the demanded augmented connectivity k. Then the Algorithm 2 with redefined condition (11) finds the $k-$CMRG $\hat{\mathcal{G}}_k^* = (\mathcal{V}, \mathcal{E}_k', \mathcal{W}_k')$ such that $|\mathcal{E}_k'| \leq (1 + \frac{1}{k})|\mathcal{E}_{opt}'|$, where $\mathcal{E}_{opt}'$ denotes the cardinality of the optimal solution required for such k, as in **Lemma** 2.*

Proof: We first prove that the solution $\hat{\mathcal{G}}_k^* = (\mathcal{V}, M' \cup F')$ from Algorithm 2 with (11) and $\mathcal{G}_k^* = (\mathcal{V}, M \cup F)$ from original Algorithm 1 have the same number of edges, if Algorithm 1 tasks as inputs the expanded graph $\mathcal{G}'$ from Algorithm 2 (both satisfy $\kappa(\hat{\mathcal{G}}') \geq k$ after graph expansion). By contradiction, we assume they have different number of edges in M' and M, namely, the following two conditions must be true at the same time.

$$\begin{gathered} \beta \cdot |M'| + \Sigma_{(v_i,v_j) \in M'}\{w_{i,j}\} < \beta \cdot |M| + \Sigma_{(v_i,v_j) \in M}\{w_{i,j}\} \\ |M'| > |M| \end{gathered} \tag{12}$$

Recall that $\beta >> 2 \cdot \Sigma_{\forall w_{i,j} \in \mathcal{W}} |w_{i,j}|$, hence it is straightforward that the two equations contradict to each other, proving that $|M'| = |M|$. Then since the computation of the inclusionwise minimal edge set is the same in both of the algorithms, we conclude that $|\mathcal{E}_k'| \leq (1 + \frac{1}{k})|\mathcal{E}_{opt}|$.□

With the minimum number of edges and total weights for the obtained $k-$CMRG, it thus invokes the least $k-$connectivity constraints that are minimally violated by the current behavior-prescribed robots controllers. The resulting $\hat{\mathcal{G}}_k^*$ therefore specifies the optimal $k-$connectivity graph $\mathcal{G}^{c*} = \hat{\mathcal{G}}_k^*$ for the given connectivity demand k to enforce in the optimization problem (8). For completeness, we provide a detailed algorithm framework of our $k-$CMRG method from Algorithm 2 in Algorithm 3.

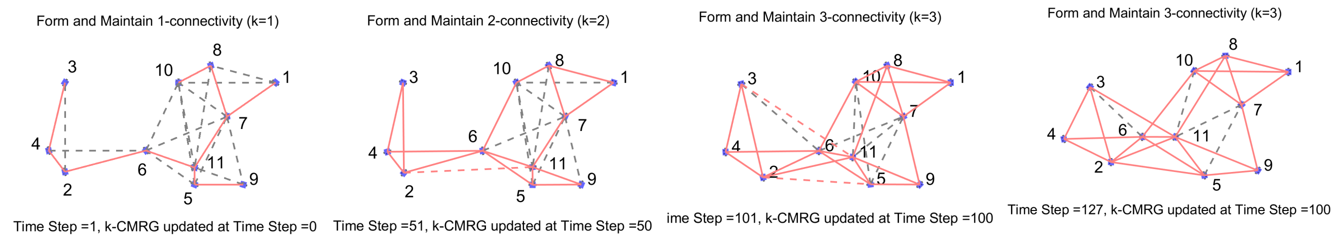
Algorithm 3 $k-$Connected Minimum Resilient Graph ($k-$CMRG)

Input: $\hat{\mathcal{G}}'(\mathcal{V}', \mathcal{E}', \mathcal{W}') \leftarrow \hat{\mathcal{G}} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, k
Output: $\hat{\mathcal{G}}_k^*$
1: **while** $\min\{\deg(v)\} < k, \forall v \in \mathcal{V}$ **do**
2: $\quad$ $\hat{\mathcal{G}}' \leftarrow$ExpandGraphOneHopNeighbor($\hat{\mathcal{G}}'$)
3: **for** all $v \in \mathcal{V}$ **do** $b(v) \leftarrow \deg(v) + 1 - k$
4: Get $b-$matching edge set: $\bar{M}' \leftarrow b-$Suitor($\hat{\mathcal{G}}', b$)
5: **while** $F' \neq \emptyset$ **do**
6: $\quad$ $M' \leftarrow \hat{\mathcal{G}}' \setminus \bar{M}'$, $F' \leftarrow \emptyset$, $\mathcal{G}_t \leftarrow \hat{\mathcal{G}}'$
7: $\quad$ **for** all $e \in \bar{M}'$ **do**
8: $\quad\quad$ $\mathcal{G}_t' \leftarrow$ CreateDigraph($\mathcal{G}_t$, unit capacities)
9: $\quad\quad$ num_disjoint_path $\leftarrow$ max_flow($\mathcal{G}_t', e_{source}, e_{sink}$)
10: $\quad\quad$ **if** num_disjoint_path$>k$ **then**
11: $\quad\quad\quad$ $\mathcal{G}_t$.remove(e)
12: $\quad\quad$ **else**
13: $\quad\quad\quad$ $F' \leftarrow F' \cup e$
14: $\quad$ $\hat{\mathcal{G}}' \leftarrow$ExpandGraphOneHopNeighbor($\hat{\mathcal{G}}'$)
15: **return** $\hat{\mathcal{G}}_k^* \leftarrow (\mathcal{V}, M' \cup F')$

From Line 1-4 in Algorithm 3, the min-size $(k-1)$ edge cover M' in (11) is obtained by first solving for its complementary edge set $\bar{M}'$ with the following condition.

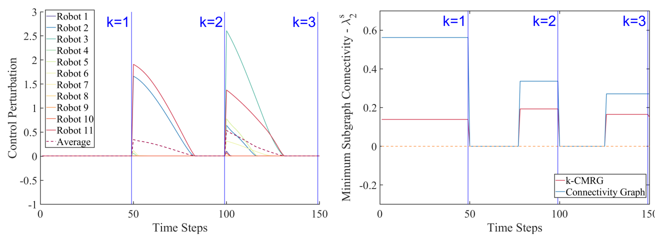
$$\begin{aligned} \bar{M}' = \underset{\bar{M}' \subseteq \mathcal{E}}{\arg\max} \quad & \beta \cdot |M'| + \Sigma_{(v_i,v_j) \in M'}\{w_{i,j}\} \\ \text{s.t.} \quad & \deg_{\bar{M}'}(v) \leq \deg(v) + 1 - k \quad \forall v \in \mathcal{V} \end{aligned} \tag{13}$$

The above problem is known as a weighted $b-$matching problem [23], [24] and we implement a subroutine $b-$Suitor



(a) Time Step = 1, k = 1 (b) Time Step = 51, k = 2 (c) Time Step = 101, k = 3 (d) Time Step = 127, k = 3 (Converged)

Fig. 2: Simulation example of 11 robots reconfigure to achieve increasing connectivity demands. Grey dash edges are real-time connectivity edges when the connected pairwise robots stay within the limited communication range. Red solid edges are the computed $k-$CMRG edges exist in the current connectivity graph. Red dashed edges are the edges of $k-$CMRG to form (not belong to the current grey connectivity graph) and thus to reach the desired connectivity.



(a) Control perterbation (b) Subgraph Algebraic Connectivity

Fig. 3: Performance of resilient connectivity maintenance with $k-$CMRG for Fig. 2. (a) Control perturbation computed by $\frac{1}{n}\sum_{i=1}^{n}\left\|u_i^*-\hat{u}_i\right\|^2$. (b) Minimum subgraph algebraic connectivity evaluated by second smallest eigenvalue of laplacian assuming $k-1$ robots being taken out. >0 means graph remains connected.

[28] to solve it efficiently. When computing for the inclusionwise minimal edge set F' in Line 5-14, we start with empty set F' and initialize the current graph to be the present connectivity graph $\hat{\mathcal{G}}$. Then each candidate edge e not in the $k-1$ edge cover set M' is checked by finding if there are at least $(k+1)$-node disjoint paths in the current graph $\mathcal{G}_t$. This is done by creating a directed graph from $\mathcal{G}_t$ and run a max flow algorithm (Line 8-9) using sub-routine from [29]. If yes, then the current candidate edge e is not critical (see [24]) and hence removed from current graph. Otherwise, the edge is critical and shall be inserted into the set F' to consist of final $k-$CMRG $\hat{\mathcal{G}}_k^*$. This comes from the fact that for an optimal $k-$CMRG with least number of edges, each edge is critical and there will be no more than $k+1$ disjoint paths between the two end nodes for the edge [24]. As mentioned, in case that $\kappa(\hat{\mathcal{G}}') \leq k$, we keep looping from Line 5 to Loop 14 and expanding the current graph, until no more critical edges are found.

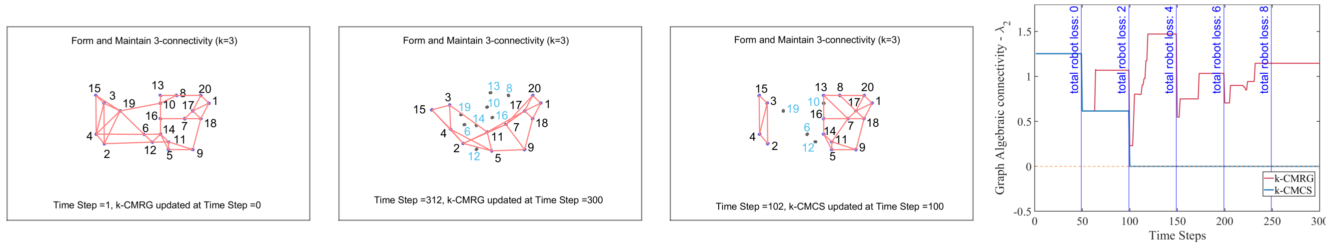
Thus, with the final $k-$CMRG $\hat{\mathcal{G}}_k^*$ obtained from our Algorithm 3 as the optimal $k-$connectivity graph $\mathcal{G}^{c*} = \hat{\mathcal{G}}_k^*$ in (9), we can specify the safety and connectivity barrier certificates (3) and (6) to invoke linear constraints and efficiently solve the original quadratic programming (QP) problem in (7). The resultant controllers satisfy safety and $k-$connectivity constraints and minimally disrupted from the original controllers.

IV. Results

To evaluate our proposed $k-$CMRG and the resilient connectivity maintenance framework, we designed three sets of experiments in simulation: i) $n = 11$ robots driven by uniform original task controller $\hat{u}_i = 0$ and to keep reconfiguring for achieving the increased connectivity demands over time, ii) $n = 20$ robots driven by the same task controller $\hat{u}_i = 0$ with desired connectivity maintenance in presence of continuous loss of robots, and iii) $n = 20$ robots tasked to perform rendezvous and dynamic circling formation around three predefined task areas, while achieving dynamic connectivity demands and staying resilient in presence of removal of robots due to failures. In all of the experiments, we are assuming limited sensing for collision avoidance, limited communication range, and bounded velocity for the robots. We apply the resilient optimization-based controller in (7) with single-integrator dynamics to the unicycle mobile robots using kinematics mapping in [27].

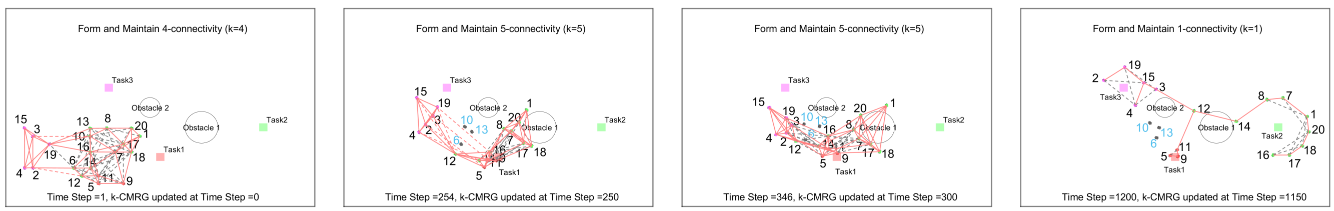
A. Reconfiguration of static robot team with increasing connectivity demands

Fig. 2 shows the simulation example of 11 robots with zero task-related control inputs and our $k-$CMRG method for connectivity enhancement. At initial configuration Fig. 2a, the robots are tasked to maintain $1-$connectivity and the $k-$CMRG returns the minimum spanning tree invoking the least number of connectivity constraints with the smallest weights, reflecting the minimum efforts required to maintain the connectivity. At time step $t = 50$ in Fig. 2b, the connectivity demand increases to $k = 2$ that is higher than the current graph connectivity. In this case, our $k-$CMRG returns a $2-$connectivity graph with one new edge between robot 2 and 11 and with such specified constraint, our resilient connectivity control framework employs FCBF to drive the robots to form the connectivity edge as shown in Fig. 2c at $t = 101$. Likewise, the new demand of $3-$connectivity invokes two more edges to form, which enforces the robots to reconfigure and quickly converge to the states with satisfying connectivity (Fig. 2d). The performance of the maintained connectivity is plotted in Fig. 3, showing the convergence of the robots after reconfiguration. In absence of actual robot removal, Fig. 3b demonstrates the algebraic connectivity of the subgraph of current connectivity graph if randomly taking



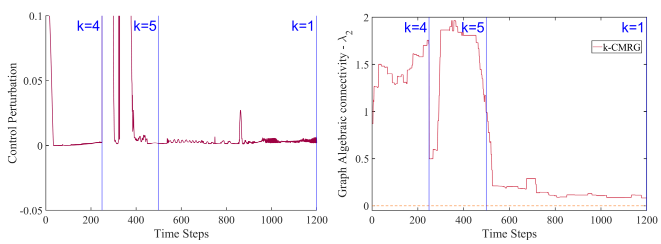
(a) Time Step $= 1$ with $k-$CMRG (b) Time Step $= 312$ with $k-$CMRG (c) Time Step $= 102$ with $k-$CMCS (d) Algebraic Connectivity

Fig. 4: Simulation example of 20 initially static robots in presence of continuous robot failures. Red edges from (a)-(b) are defined by computed $k-$CMRG and robots with cyan index indicates the faulty robots that are no longer involved in the connectivity graph. Red edges in (c) shows the failure cases of robust connectivity maintenance method [15] due to the lack of resilience consideration. (d) plots the actual algebraic connectivity λ_2 of the real-time connectivity graph from $k-$CMRG and $k-$CMCS [15]. Connectivity preserves if $\lambda_2 > 0$. It shows $k-$CMRG (red curve) is able to keep the graph connected and recover the connectivity in presence of loss of robots).



(a) Time Step=1 with $k = 4$ (b) Time Step=254 with $k = 5$ (c) Time Step=346 with $k = 5$ (d) Time Step=1200 with $k = 1$

Fig. 5: Simulation example of 20 robots executing multiple behaviors with changing connectivity demands and robot failures: green robots 1, 6, 7, 8, 10, 12, 13, 14, 17, 18, 20 and magenta robots 2, 3, 4, 15, 19 are tasked to circle around task 2 and task 3 area respectively. Red robot 5, 9, 11 are tasked to rendezvous towards task 1 area. Grey dash edges are real-time connectivity edges. Red solid edges are the computed $k-$CMRG edges exist in the current connectivity graph. Red dashed edges are the edges of $k-$CMRG to be formed by the robots for increased connectivity.



(a) Control perturbation (b) Algebraic connectivity

Fig. 6: Performance of $k-$CMRG from simulation example in Fig. 5. Control perturbation quickly converges to zero due to minimally disruptive connectivity maintenance from $k-$CMRG and network connectivity is always satisfied and recovered despite robot failures.

out $k-1$ robots. As seen from the figure for $t = 50-60$, multi-robot network will get disconnected if removing one robot from the currently 1-connected graph. After achieving the connectivity $k = 2$ at $t = 60$, the resilience of the network is improved and hence robust to the removal of the robots. Note that the robots remain collision-free due to the employed safety barrier certificates in (3).

B. Connectivity Enhancement in presence of continuous loss of robots

One of the advantage of the proposed $k-$CMRG is to enable increased connectivity over time for resilient robot team under faulty situation, as shown in Fig. 4. In this experiment, 2 randomly selected robots will stop connecting other robots every 50 time steps starting from $t = 50$ (Fig. 4d). The robots are tasked to maintain $k = 3$ connectivity in presence of robot losses. With the proposed $k-$CMRG, the connectivity of multi-robot network is preserved and actively recovered even if robots keep failing with a total of 8 failing robots (Fig. 4b). In comparison, we implemented the robust connectivity approach [15] that seeks to preserve the robust connectivity of the current graph. As shown in Fig. 4c and Fig. 4d, although the robotic team could stay connected with the removal of a few robots, the robot team is not able to recover the decreased connectivity over time and hence gets disconnected eventually.

C. Reconfiguration of moving robot team with obstacles, robot failures, and changing connectivity demands

In this task, 20 robots have been divided into 3 subgroups and each performing an individual behavior with the $k-$CMRG in presence of 2 static obstacles and robot failures. The connectivity demands are randomly chosen to be $k = 4$ for $t = 0-250$, $k = 5$ for $t = 250-500$ and $k = 1$ for $500-1200$. Robot 6, 10 and 13 are removed at $t = 250$ in Fig. 5b to simulate faulty situation. As shown in Fig. 5d, the goal for magenta robots and green robots are to circle around assigned task area 3 and 2 respectively, while red robots are to rendezvous to red task 1 area. Without connectivity maintenance, the robot team could get disconnected easily. Robots start from Fig. 5a to reconfigure and achieve the higher demand of connectivity $k = 4$ while executing their original behaviors (the control error reduced to almost zero quick after $t = 0$ as shown in Fig. 6a). At $t = 250$, the connectivity demand further increases to $k = 5$ and three robots are lost, resulting in largely reduced network connectivity as observed in Fig. 6b. The $k-$CMRG is able to quickly reconfigure the robots to reach the desired connectivity as shown in Fig. 5c, where all $k-$ CMRG edges are established as solid red edges. Meanwhile, the original behaviors of the robots are preserved as the control perturbation to the original controller reduced to almost zero soon after $t = 400$. And with the decreased connectivity demand to $k = 1$ after $t = 500$, the robots are able to stay as close to their tasks while ensuring the required connectivity in Fig. 5d, demonstrating the flexibility of our resilient connectivity maintenance method with $k-$CMRG.

V. CONCLUSION

In this paper, we propose resilient connectivity maintenance algorithms to ensure minimally disruptive connectivity enhancement for multiple robots during the execution of their primary tasks. In particular, we propose a $k-$Connected Minimum Resilient Graph ($k-$CMRG) to allow for reconfiguration of the multi-robot system to provably achieve any connectivity demand, while ensuring the robot original behaviors are minimally modified in the context of quadratic programming with the control barrier fucntions (CBF) and Finite-Time Convergence Control Barrier Functions (FCBF). Such algorithms improves the resilience of connectivity reduction for open robot team with continuous robots arrival and removal. Several simulation examples are demonstrated to validate our algorithm in various challenging scenarios. Future work includes the decentralization of the resilient connectivity controller and the real-world implementation in uncertain environments. We will also extend our approach by considering more complicated communication models.

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